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Probabilistic generation of quantum contextual sets

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ABSTRACT

We give a method for exhaustive generation of a huge number of Kochen–Specker contextual sets, based on the 600-cell, for possible experiments and quantum gates. The method is complementary to our previous parity proof generation of these sets, and it gives all sets while the parity proof method gives only sets with an odd number of edges in their hypergraph representation. Thus we obtain 35 new kinds of critical KS sets with an even number of edges. We also give a statistical estimate of the number of sets that might be obtained in an eventual exhaustive enumeration.

1. Introduction

Quantum contextuality is the property of a quantum system that a result of any of its measurements might depend on other compatible measurements that might be carried out on the system. The so-called Kochen–Specker (KS) sets provide constructive proofs of quantum contextuality and therefore provide straightforward blueprints for their experimental setups. KS sets are likely to find applications in the field of quantum information, similar to ones recently found for the Bell setups in implementing entanglements [1,2]. A. Cabello's result [3], according to which local contextuality can be used to reveal quantum nonlocality, supports the conjecture. Also our results [4–6] show that KS sets play an important role in Hilbert space description of complex setups.

A series of KS experiments have been carried out in the last ten years. The most recent ones made use of quantum gates and employed recently developed quantum information techniques of handling, manipulating, and measuring of qubits by means of quantum circuits of such gates. The experiments were proposed, designed, and carried out for spin-$\frac{1}{2}\otimes\frac{1}{2}$ particles (correlated photons or spatial and spin neutron degrees of freedom) [7–17]. The KS sets that were used in these experiments were from $2\times 2=4$-dim Hilbert space. In particular they were either from the 24–24 class of KS sets (set with 18 through 24 vectors and 9 through 24 orthogonal vector tetrads) or the Mermin set [18].

Therefore in [19,18] we exhaustively generated all KS sets from the 24–24 class without ascribing coordinates to Hilbert vector (states, wave function) components. That was done by means of McKay–Megill–Pavicic (MMP) hypergraph representation (MMP diagrams). Such sets can be implemented directly in both 3-dim (spin-1, qutrits) and 4-dim (spin-$\frac{3}{2}$) KS setups by means of, e.g., generalized Stern–Gerlach devices [20].

Most recently [21,22] we generated a number of KS sets from a 4-dim 60–75 KS set we obtained from the so-called 600-cell (the 4-dimensional analog of the icosahedron) [23]. Since they all stem from this single 60–75 set and since no set from the 24–24 class belongs to it, we call it the *60–75 KS class*. The experimental implementations of the sets belonging to this class are straightforward although demanding. For instance, we let a spin-$\frac{3}{2}$ systems through a series generalized Stern–Gerlach devices, enabling control over outcoming directions of particles [20]. The approach can also be used to make quantum gates.

For any experimental application it is not viable to consider all possible millions of sets but only those that can be experimentally distinguished. Hence, we extract critical non-redundant non-isomorphic KS sets with 26 to 60 vectors from all possible 60–75 KS sets. "Critical" means that they are minimal in the sense that no orthogonal tetrads can be removed without causing the KS contradiction to disappear. We found several thousand critical KS setups that have no experimental redundancies.

In [22] we developed a method of generation of all those KS sets that allow the so-called parity proofs (see below). However,

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the parity proofs are applicable only to the sets with an odd number of tetrads of orthogonal vectors, and the aforementioned generation gives only such sets.

In this Letter, we describe a method for generating all KS sets from the 60–75 KS class, in particular those sets that we cannot obtain by our parity-proof-generation method. While in principle the method is exhaustive, a full generation is at present too demanding. Instead, we used random samples of the search space and applied Bernoulli trial probability analysis to obtain expected means and confidence intervals. We obtained these samples using techniques of graph theory. Also, since the parity proof method is faster for obtaining critical KS sets with odd number of vector tetrads, we concentrate on even numbers of tetrads (see Table 1). In this sense our probabilistic generation method and our parity-proof-generation method are complementary. The former method is better at finding a large number of critical sets of both kinds.

We develop algorithms that allow us to go survey a huge number of possible tetrads. In addition, based on statistical extrapolation from our sample, we give an estimation method according to which there might be $\approx 4.3 \cdot 10^{12}$ (see Fig. 3) of non-isomorphic critical KS sets that are subsets of the 60–75 set. The method is however essentially classical, so that a future exhaustive generation might give far less numbers of critical sets. If it does, it will be a measure quantum-classical difference. If not, then we will have a powerful tool for estimating the reliability of random generation of critical KS sets.

Finally, we give an overall picture of the critical KS sets we found, describe patterns we have observed in their distribution, and list some open questions about whether others that we haven't found yet exist and whether, for some sizes, we have exhausted all possible isomorphism classes.

We make use of theory and algorithms from several disciplines: quantum mechanics, lattice theory, graph theory, and geometry. Thus in the context of our study, the term "vertex" is synonymous with the terms "ray", "atom", "1-dim subspace", and "vector" that appear in the literature; "edge" with the terms "basis", "block", and "tetrad (of mutually orthogonal vectors)"; and "MMP hypergraph" with the terms "MMP diagram" and "KS sets".

2. Results

The *Kochen–Specker* (KS) theorem states that a quantum system cannot in general possess a definite value of a measurable property prior to measurement, and quantum measurements (essentially detector clicks) carried out on quantum systems cannot always be ascribed predetermined values (say 0 and 1). This means that two measurements of the same observable of the same system sometimes must yield different outcomes in different contexts. This is called the *quantum contextuality*. One way of proving the theorem is to prove the existence of KS sets, i.e., to provide algorithms for their constructive generation. The more abundant they are, the more important the contextuality of quantum mechanics appears to be.

Every KS set is a proof of the KS theorem. Kochen–Specker (KS) *set* is a set of vectors $|\psi_i\rangle, |\psi'_i\rangle, \ldots$ in $\mathcal{H}^n$, $n \geqslant 3$ to which it is impossible to assign 1's and 0's in such a way that:

1. No two orthogonal vectors are both assigned the value 1;
2. Not all of any mutually orthogonal vectors are assigned the value 0.

KS subsets of mutually orthogonal vectors in a 3-dim space we call triads, in a 4-dim space tetrads, etc. A KS set is a union of such triads, tetrads, etc. They can be represented by means of MMP hypergraph defined below. In a KS set, the vectors correspond to vertices and the tetrads to edges of MMP hypergraphs. (In the last paragraph of this page we give a pedestrian introduction to this correspondence.)

We define MMP hypergraphs as follows [19]

(i) Every vertex belongs to at least one edge;
(ii) Every edge contains at least 3 vertices;
(iii) Edges that intersect each other in $n-2$ vertices contain at least n vertices.

This definition enables us to formulate algorithms for exhaustive generation of MMP hypergraphs. We work with subsets of the starting hypergraph, the 60–75 one, so the job of generating the hypergraphs amounts to a creation of all possible subsets of the 60–75 set with a specified number of edges deleted. The "only" difficulty we face is the shear size of these generated subsets—we are dealing with a haystack of 2^{75} or 38 sextillion subsets, in which we wish to find certain "needles" i.e. critical KS sets.

The hypergraphs we obtain reflect only the orthogonal structure of KS sets and do not in any way refer to the vector components of the original 60–75 KS set. This is yet another aspect in which the present method differs from the parity-proof method we used in [22], which relies on the vector components of the vectors in each KS set that was inherited from the original 60–75 set. For each hypergraph we can, however, find appropriate vector components with our program `vectorfind` or by interval analysis we developed in [19]. These components need not be those of the vectors from the 60–75 set.

We encode MMP hypergraphs by means of alphanumeric and other printable ASCII characters. Each vertex is represented by one of the following characters: `1 2 3 4 5 6 7 8 9 A B C D E F G H I J K L M N O P Q R S T U V W X Y Z a b c d e f g h i j k l m n o p q r s t u v w x y z ! " # $ % & ' ( ) * - / : ; < = > ? @ [ \ ] ^ _ ` { | } ~`, and then again all these characters prefixed by '+', then prefixed by '++', etc.

Each edge is represented by a string of characters that represent vertices. Edges are separated by commas. All edges in a line form a representation of a hypergraph. The order of the edges is irrelevant. We often present them starting with edges forming the biggest loop to facilitate their possible drawing. The line must end with a full stop. Skipping of characters is allowed.

In Fig. 1 we show a graphical representation of the minimal (26–13) critical KS hypergraph we found and which we shall now use to show a correspondence between the vector and the MMP hypergraph representation of any KS set.

Each vertex represents a vector in a 4-dim space. For instance, $\mathtt{G} = \{1,0,0,0\}$, $\mathtt{F} = \{0,1,0,0\}$, $\mathtt{E} = \{0,0,1,0\}$, $\mathtt{D} = \{0,0,0,1\}$. They are mutually orthogonal and that means they form an edge—`GFED`. Our program `vectorfind` can assign all vectors, that correspond to edges from the 26–13 set, component values from the set $\{0, \pm 1, \pm(\sqrt{5}+1)/2, \pm(\sqrt{5}-1)/2\}$ and that means that the system of equations that define all orthogonalities for the 26–13 does have a solution. Now our program `states01` (which exhaustively checks all possible assignments) checks whether all the vertices can be ascribed 0 and 1 according to the KS rules 1 and 2 above and verifies that it is not possible. The main point here is that we can always go from MMP hypergraphs to vectors and back and that `states01` works with MMP hypergraphs. MMP hypergraphs are linear while the system of equations describing mutual orthogonality of vectors are nonlinear. Therefore the evaluation of MMP hypergraphs by means of `states01` is exponentially faster than solving nonlinear equations and this is what makes our generation of KS sets feasible in particular for those ones with an even number of tetrads (edges). While the algorithm used in `states01` is comparatively fast, the verification of KS sets for MMP hypergraphs

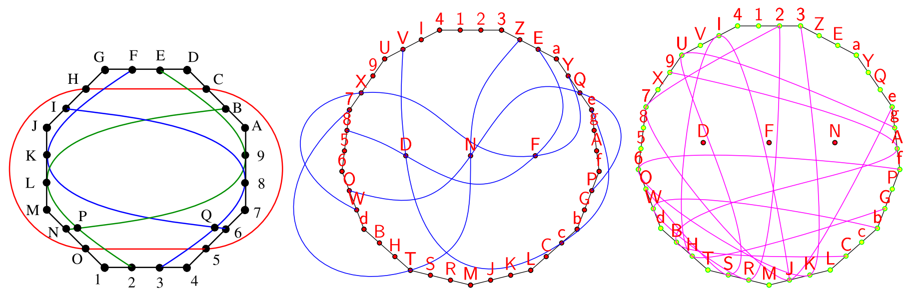

Fig. 1. The first KS set 26–13 with odd number of edges on the left vividly illustrate the parity proof: all vertices share two edges. The two graphs on the right represent 42–24 KS with some edges shown in the first figure and come in the second. They are drawn by Asymptote (vector graphic language). By changing parameters one can interactively control and change the shape of each edge line and unambiguously discern vertices that share an edge.

with an odd number of edges can be even faster using our "parity proofs" algorithm described in [22].

Parity proof. Looking at the 26–13 KS set in Fig. 1, we see that we cannot ascribe values 0 and 1 to all vertices so that in each edge we ascribe 1 to one of the vertices and 0 to the others. For in all these hypergraphs, each vertex shares exactly two edges, so there should be an even number of 1s. At the same time, each edge must contain one 1 by definition, and since there are an odd number of edges, there should be an odd number of 1s—a contradiction. □

In this parity proof and in more involved ones in [22], we have an odd number of edges. We found 90 kinds of such critical KS sets starting with a 26–13 and ending with a 60–41. The KS sets we found by means of parity proofs in Ref. [22], but did not occur in the statistical samples we used for the algorithms described in this Letter, we indicate by "⊗" in Table 1 below.

In Table 1 and Fig. 3, we show the distribution of each kind we found by the random sampling; on average $3 \cdot 10^8$ for each of 13–63 edges (1–12 and 64–75 were exhaustively scanned), we obtained 35 kinds of critical KS sets with an even number of edges, which cannot be obtained by the parity-proof method. Three examples of them are given in Figs. 1 and 2—hypergraphs automatically drawn by our program written in Asymptote.

We also obtained 62 kinds of critical KS sets with an odd number of edges. They are all among 90 kinds of KS critical sets we obtained by the parity-proof method in [22]. Those that we did not obtain in our samples are indicated by "⊗" in Table 1. Our scanning in [22] was designed to obtain as many different kinds of critical sets as possible. So, we always stopped scanning as soon as we found a new kind and therefore we cannot estimate their numbers.

In our ASCII presentation of MMP hypergraphs below we first write down n edges (tetrads) form n-gons but in general they can be written in any order. We obtained them by our program loop-big. Additional examples of hypergraphs of each kind not given here are listed in Appendix B.

There are three types of edges in an MMP hypergraph

Polygon edges those that form the n-gon.
Free edges those that contain vertices that do not belong to the n-gon. We call these vertices *free vertices*.
Span edges all others.

To better discern the vertices in an MMP hypergraph we often represent them by two figures showing first free edges and then span edges.

42–24 (13) 3124, 4VIU, UX97, 7586, 6WOd, dHBT, TRSM, MKJL, LcCb, bPGf, fgAe, eYQa, aZE3, 9ABC, DEF8, GHIJ, NOPQ, WXYF, ecVD, gUSO, ZTN7, bWR2, fK63, eJ72. Shown in Fig. 1.

50–30 (15) 3124, 4DEF, Fm6i, ihbP, POQJ, JHIG, GoCj, jkKS, SRTU, UeLd, dl7W, WVNX, Xg8f, fnAZ, ZYa3, 5678, 9ABC, KLMN, bcaM, TQFC, ecEB, lkPA, mdZO, mgRI, iYKH, njcW, jhg4, VHA4, oaR7, oife. Shown in Fig. 2.

60–40 (18) 3124, 4576, 6yau, uvVt, trqs, soTP, Pxh9, 98AB, BpWM, MJLK, KicU, UwOl, lmnk, kjSH, HGIF, FCED, DYRf, feg3, NOPM, QRSB, TUVW, XYZW, abcd, hiI2, odZS, pjc7, qLA6, wtbH, xvpg, yxnR, rhSJ, vOEA, mYP6, ieCB, oneG, ncXA, ulhf, reaO, wpoD, slB4. Shown in Fig. 2.

To obtain these hypergraphs, we used a procedure that strips one edge at a time. For n input hypergraphs each with b edges, $n \cdot b$ output hypergraphs, each with $n - 1$ edges, were generated. After passing these output hypergraphs through several filters to eliminate unconnected hypergraphs, duplicates, non-KS sets, and isomorphic sets, a smaller number of hypergraphs usually resulted. To keep the run time feasible, we took a semi-random[1] sample of the generated hypergraphs so that in the end we would have approximately the same number n of KS sets to send to the next edge-stripping step. The algorithms and programs we used to obtain KS critical sets listed in Table 1 are described in Section 3.

3. Algorithms

For the purpose of the KS theorem, the vertices of an MMP hypergraph are interpreted as rays, i.e. 1-dim subspaces of a Hilbert space, each specified by a representative (non-zero) vector in the subspace. The vertices on a single edge are assumed to be mutually orthogonal rays or vectors. In order for an MMP hypergraph to correspond to a KS set, first there must exist an assignment of vectors to the vertices such that the orthogonality conditions specified by the edges are satisfied. Second, there must not exist an assignment of 0/1 (non-dispersive or classical) probability states to the vertices such that each edge has exactly one vertex assigned to 1 and others assigned to 0.

For a given MMP hypergraph, we use two programs to confirm these two conditions. The first one, vectorfind, attempts to find an assignment of vectors to the vertices that meets the above requirement (see Appendix B). The second program, states01 (see Appendix A), determines whether or not a 0/1 assignment is possible that meets the above requirement. An additional option was added to states01 to determine if a hypergraph is critical, i.e., whether the hypergraph is a KS one but becomes non-KS if any single edge is removed.

The 60-vertex, 75-edge MMP hypergraph based on the 600-cell described above (which we refer to as 60–75) has been shown to

[1] "Semi-random" because while we removed random edges from each input hypergraph, we chose uniformly spaced samples from the set of input hypergraphs (see 1st paragraph of Appendix C).

Table 1
List of KS critical sets we obtained in this Letter. By ⊗ we indicate the existence of KS critical sets (at least one set) we obtained in [22] by the parity proof method. The average sample sizes of $3 \cdot 10^8$ sets used here were too small to obtain them by our algorithms/programs.

	Critical KS sets with odd number of edges															…with even number of edges								
ver	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	24	26	28	30	32	34	36	38	40
26	1																							
30		3																						
32			1																					
33			2																					
34			5																					
36				11																				
37				9																				
38				6	10																			
39					30																			
40					38	10																		
41					22	5																		
42					6	16	3									1								
43						22	38																	
44						14	16	⊗																
45						3	5	32									2							
46						1	3	130	⊗									3						
47							1	74	⊗									6						
48							2	19	9	⊗								11						
49							⊗	⊗	11	1								3	9					
50							1	⊗	7	13									39					
51								⊗	1	33	⊗								19	18				
52								⊗	⊗	37	33								4	69				
53								⊗	⊗	11	114	⊗							1	73	45			
54								⊗	⊗	⊗	153	16								26	275			
55									⊗	1	56	158	⊗							5	339	25		
56									⊗	⊗	21	241	28								136	262		
57										⊗	1	133	378								54	448	45	
58										⊗	⊗	30	678	27							2	256	493	
59											⊗	2	308	381							1	55	864	16
60											⊗	⊗	48	562	1							5	316	145

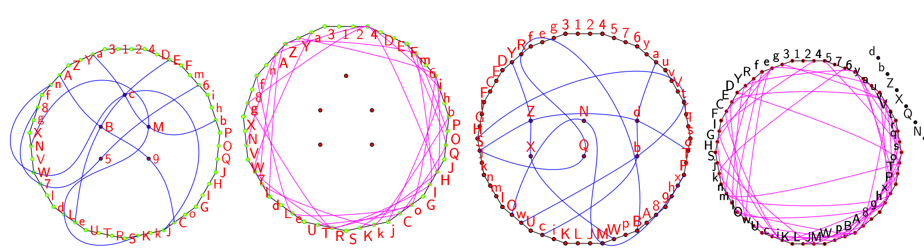

Fig. 2. The two left figures represent a critical 50–30 KS set (15-gon): the first shows free edges only and the second span edges only. The right two figures represent a critical 60–40 KS set (18-gon) (one of the biggest critical KS sets we found): the first shows free edges; the second, span edges. Letters and edges might appear overcrowded, but the MMP notation provides a clear alternate representation for each of them—we need not give ASCII characters at all. (We give enlarged figures in Appendix B.)

be a KS set [24]. However, it has redundancies (is not a critical set) because we can remove edges from it and it will continue to be a KS set. The purpose of this study was to try to find subsets of the 60–75 hypergraph that are critical i.e. that are minimal in the sense that if any one edge is removed, the subset is no longer a KS set.

While the program `vectorfind` independently confirmed that 60–75 admits the necessary vector assignment, such an assignment remains valid when an edge is removed. Thus it is not necessary to run `vectorfind` on subsets of 60–75. However, a KS set will eventually become a non-KS one when enough edges are removed, and the program `states01` is used to test for this condition.

A basic method in our study was to start with the 60–75 hypergraph and generate successive subsets, each with one or more edges stripped off of the previous subset, then keep the ones that stayed KS and discard non-KS one. Of these, ones isomorphic to others were also discarded.

The program `mmpstrip` was used to generate subsets with edges stripped off. The user provides the number of edges k to strip from an input MMP hypergraph with n edges, and by default the program will produce all $\binom{n}{k}$ subsets with a simple combinatorial algorithm that generates a sequence of subsets known as the "banker's sequence" [25]. Partial output sets can be generated with start and end parameters. By default, `mmpstrip` will scan linearly through the edges to pick every ith one when the increment parameter is i. The program will optionally randomise the edge selection, so that while a fraction $1/i$ of edges is picked from each input hypergraph, which edges are picked are random.

Optionally, `mmpstrip` can take truly random samples with replacement (for a given number of edges) from the starting 60–75 MMP hypergraph (in contrast to the semi-random method of the previous paragraph). This mode was used to verify or improve some of the statistical estimates in Fig. 3. A cryptographic hash of the time of day, process ID, and CPU time is used as the seed for the pseudo-random number generator. The seed may also be provided by the user in order to repeat a result.

In order to detect isomorphic hypergraphs, one of two programs was used. For testing small sets of hypergraphs, we used the program `subgraph` described in Ref. [18], which has the advantage

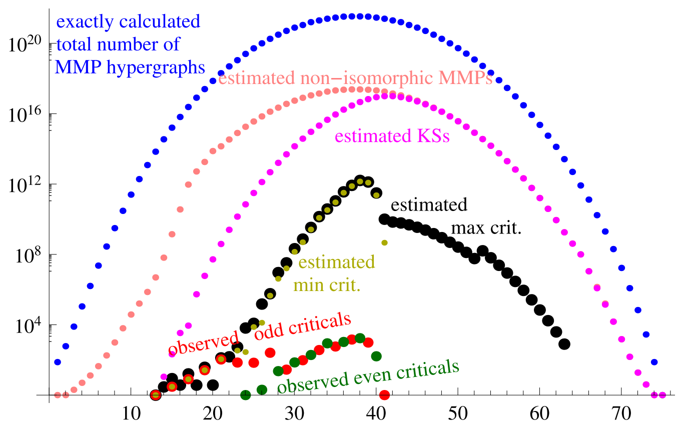

Fig. 3. Overall statistics calculated for subsets of 60–75 given on a logarithmic scale. The samples (for 13–63 edges; 64–75 search was exhaustive) contained on average $3 \cdot 10^8$ MMP hypergraphs. "Observed odd (even) criticals" refer to odd (even) numbers of *critical* KS sets. The sudden jump in the "estimated max crit". (EMC) plot at 53 edges is caused by a change in the sample size, as explained in the text. The EMC points are not plotted when zero.

of displaying the isomorphism mapping for manual verification. For a large number of hypergraphs, we used Brendan McKay's program `shortd`, which has a much faster run time.

The program `longest` singles out longest loops from the list of all possible loops (which is the output of the program `loopbig`). The program `parse` or `parse_all` then "writes" a program or programs in the vector graphics language `Asymptote` for drawing a chosen hypergraph or all hypergraphs, respectively.

The longest loop of each hypergraph is drawn as n-sided regular (equilateral and equiangular) polygon, where n is the number of edges in the loop. By default, free vertices, i.e. vertices that are not on the loop, are placed inside the polygon, off-centre, on vertical lines, with not more than 4 vertices on one line, but the user can change options for their placement. Edges contained in the longest loop are drawn as straight lines, while other edges are drawn as Bézier curves (specifically, `Asymptote` is based on Donald Knuth's `METAFONT`). The user can interactively change the "tension" of the curve and the amount of "curl" at its endpoints, in order to interactively control and change the shape of each edge line and unambiguously discern vertices that share an edge.

4. Sample space statistics

There are $3.8 \cdot 10^{22}$ possible subsets of the 60–75 set (disregarding any symmetry) and, among them, approximately $7.5 \cdot 10^{17}$ KS sets. An exhaustive search for critical KS sets was not feasible for the present survey, but it may become feasible in the future, possibly requiring a year or more on a supercomputer.

For our survey, we searched a total of around 10^{10} KS sets, randomly chosen for a given edge size, to find the critical KS sets among them. We then performed a statistical analysis to estimate the total number of critical KS sets that would be found by an exhaustive search. The final result is that we can expect a total of $4.3 \cdot 10^{12}$ non-isomorphic critical sets, with a 95% confidence interval between $4.0 \cdot 10^{12}$ and $4.6 \cdot 10^{12}$ based on the statistical model we used.

If an exhaustive search is eventually performed, it is possible to store the complete set of non-isomorphic critical sets with current technology. Each critical set (in MMP hypergraph notation) requires an average of about 260 bytes, thus requiring $260 \cdot 4.3 \cdot 10^{12} = 1.1 \cdot 10^{15}$ bytes (1.1 petabytes) of storage.

The plots of Fig. 3 provide an overview of the subsets of 60–75, broken down by the number of edges. These plots are intended to provide a guideline for estimating the work that would be required for an exhaustive search for a particular number of edges or range of them. The details of the statistical methods we used and a detailed description of Fig. 3, are given in Appendix C.

In our survey, no critical sets were observed in KS sets with 42 or more edges. In these cases, EMC in Fig. 3 has little to do with the number of critical sets (if any) that actually exist in that edge range. Instead, it could be interpreted as "zero with statistical noise" and is primarily a function of the number of samples we took and the search space size for the particular edge size. It simply indicates that, based on a Bernoulli trial probability model, it is unlikely (with 95% confidence) that there are *more* critical sets than EMC The sudden jump between 52 and 53 edges is due to the fact that we changed the number of samples from $5.3 \cdot 10^7$ per edge size to $8.6 \cdot 10^6$. If an exhaustive search shows that there are no critical sets at all above 41 edges, that will be completely consistent with the EMC bound. In fact, we conjecture that the number of critical sets will be zero soon after 41 edges (see Appendix C, last paragraph).

5. Conclusions

Kochen–Specker (KS) sets and setups proposed, designed, and experimentally carried out so far were either 3-, 4-, 8-, ... dimensional KS sets (Peres', Cabello's, etc.) or the Mermin set. They aim at finding particular valuation of the KS observables that prove the quantum contextuality and disprove any noncontextual classical valuations of those observables. Our aim is to make KS sets independent of a particular choice of either vectors or observables so as to make them suitable for building quantum gates in quantum circuits.

For this application, we should have a choice of gates of different sizes, that is, consisting of sufficiently many vectors and sufficiently many gates for a chosen number of vectors, and this is what we achieved in the previous sections. We generated a large number of 4-dim critical non-redundant non-isomorphic KS sets with 26 to 60 vectors based on the 600-cell (the 4-dimensional analog of the icosahedron). "Critical" means that no orthogonal tetrads can be removed without causing the KS contradiction to disappear. In other words, they represent a KS setup that has no experimental redundancy.

The generation was achieved by algorithms and computer programs described in Section 3, with which we found the critical sets summarized in Fig. 3. In Section 4 and Appendix C, we give the detailed statistical estimates of the total critical sets that exist based on our samples. The statistical techniques we used are general-purpose and can be useful for any similar experiment in which an exhaustive enumeration of outcomes is not feasible.

Critical sets obtained by a future exhaustive generation might be far less numerous then their statistical estimates given above. We could not make any realistic predictions on numerosity and existence of critical sets that might be observed in the future but have not been observed so far. Our estimates of the total number of KS sets is in good agreement with the data we so far obtained by exhaustive generation. E.g., by exhaustive generation of KS sets with 63, 64, 65, and 66 edges we obtain $1.8 \cdot 10^9$, $4.1 \cdot 10^8$, $1.0 \cdot 10^8$, and $1.1 \cdot 10^7$ sets versus estimated $1.8 \cdot 10^9$, $3.4 \cdot 10^8$, $5.7 \cdot 10^7$, and $8.8 \cdot 10^6$, respectively.

The main theoretical results of our generation are that

- the 24–24 and the 60–75 classes are disjoint (in the sense that the biggest set of the 24–24 class is the single Peres' 24–24 set and the smallest set from the 60–75 class is the 26–13 one;
- the maximal loop of all sets from the 24–24 class is always a hexagon while the maximal loops of the sets from the 60–75 class grow (form at least an octagon) as the number of vectors and edges increase (see Figs. 1–2);

- there is an unexpectedly large and rich universe with an estimated $4.3 \cdot 10^{12}$ non-isomorphic critical sets inside of the 60–75;
- in [4] we found that only one of the known 3-dim KS sets passes a series of equations (but not all) that hold in any Hilbert space—the so-called orthoarguesian equations. We have not found any such KS set in the 60–75 class so far. Both results show that orthogonality of vectors does not suffice for a complete Hilbert space description of KS sets;
- there is only one KS set with 24 vectors (vertices) and 24 tetrads (edges), and it contains all KS sets from the 24–24 class with the chosen values of vector components [18]. In contrast to this, there are many non-isomorphic KS sets with 60 vectors and 60 tetrads with many non-isomorphic subsets.

Another open question is to find physical and geometrical reasons for having only hexagon maximal loops in the 24–24 class and for having particular octagons, nonagons, decagons, etc., in the 60–75 class.

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Appendix A. Algorithms and programs behind Table 1

The iterative processes and algorithms and programs we used to obtain the critical KS sets listed in Table 1 are given as a supplementary material.

Appendix B. Samples of KS set with even number of edges

ASCII MMP hypergraphs for critical KS sets with even number of edges, additional figures, and a detailed vector-hypergraph correspondence are given as a supplementary material.

Appendix C. Details for sample space statistics

Theoretical details that served us to obtain Fig. 3 are given as a supplementary material.

Appendix D. Supplementary material

Supplementary material related to this article can be found online at doi:10.1016/j.physleta.2011.07.050.

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