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Minimal Quantum Logic with Merged Implications

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It is shown that the property of orthomodularity can be interpreted as a particular reduction of the operations of implication to the relation of implication, and a physical interpretation of the result is given.

1. INTRODUCTION

Soon after quantum logic was established as a calculus of quantum mechanics in the mid-1960s it became clear that the sense in which the calculus can be considered as a proper logic was yet to be established. The missing element for such a characterization was a natural deduction scheme and an appropriate notion of implication, on the one hand, and the absence of simple semantics, on the other.

As for the former problem, several lines of investigation have been carried out in the meantime, but a univocal answer has not been found. The approach adopted, e.g., by Goldblatt (1974) and Nishimura (1980) employs nothing but a *relation* of implication, which some authors hold to be perhaps the only proper implication (Greechie and Gudder, 1973) [and which is also called a formal or semantical implication (Hardegree, 1981a)]. Notwithstanding this opinion, a number of authors developed systems employing one of the possible *operations* of implication of the object language as defined by means of the other operations (conjunction, disjunction, and negation) (Finch, 1970; Clark, 1973; Kalmbach, 1974; Abbott, 1976; Hardegree, 1981a–c). It has been shown as well that several quantum logical systems can be obtained by adding an independent operation of implication to the object language of the considered calculus (Kron *et al.*, 1981).

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As for the semantics of quantum logic, the standard semantics (truth polyvaluation included) amounts to the mapping of its propositions onto the elements of an algebraic structure and therefore does little for the solution of the problem of whether quantum logic is to be considered a proper logic. On the other hand, the more genuine, Kripkean semantics is known to be applicable in a standard way (i.e., using a first-order characterization of a relation which determines the appropriate Kripke frame) only to minimal quantum logic (Dishkant, 1972; dalla Chiara, 1976) [also called orthologic (Goldblatt, 1974)]. Namely, it turns out that the property of orthomodularity, or, more precisely, the so-called axiom of orthomodularity, when added to minimal quantum logic in order to give quantum logic, destroys the possibility for Kripke's semantics of the first order to be ascribed to quantum logic (Goldblatt, 1984). Since an appropriate notion of implication is indispensible for a realization of Kripke's semantics, it is of interest to find out how the orthomodularity is connected with the role of implication operations in the deduction scheme of quantum logic. When we check whether the *operations* of implications, definable in the object language of quantum logic and reducible to the "classical" implication for commensurable propositions, "work" in the same "minimal" way in which the *relation* of implication "works" in minimal quantum logic, we find out that all but one (the "classical" implication itself) of them do (Hardegree, 1981a). Does this mean that the property of orthomodularity and a particular reduction of these operations of implication to the relation of implication amount to the same thing? Yes, it does. And it is the purpose of this paper to prove it.

2. QUANTUM LOGIC, MINIMAL QUANTUM LOGIC, AND CLASSICAL LOGIC

Our main concern while defining the logics will be to avoid a deduction scheme based on the operation of implication, employing the relation of implication instead. To achieve this aim, we shall adopt Ackermann's (1956) schemata, adapted by Kotas (1971) and Goldblatt (1974). Moreover, we are going to consider the relation of implication between two propositions A and B, denoted by $A \vdash B$, and called a "scheme" [the analogue to Kotas' (1971) "logical scheme"], an element of the object language, thus stressing the syntactic role the relation of implication can play in the deduction scheme of the logic, up to the limitation that it cannot be nested.

The propositions are based on elementary propositions $p_0, p_1, p_2, \ldots$, and the following connectives: $\neg$ (negation; unary connective) and $\wedge$ (conjunction; binary connective).

The set of propositions Q is defined formally as follows:

p_j is a proposition for $j = 0, 1, 2, \ldots$

$\neg A$ is a proposition iff A is a proposition
$A \wedge B$ is a proposition iff A and B are propositions

The disjunction connective is introduced by the following definition:

$$A \vee B := \neg(\neg A \wedge \neg B)$$

The operations of implication are introduced as follows:

$A \to_0 B := \neg A \vee B$ (classical implication)
$A \to_1 B := \neg A \vee (A \wedge B)$ (Sasaki, Mittelstaedt, or ortho-)
$A \to_2 B := B \vee (\neg A \wedge \neg B)$ (Dishkant)
$A \to_3 B := (A \wedge B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B)$ (relevance)
$A \to_4 B := (A \wedge B) \vee (\neg A \wedge B) \vee ((\neg A \vee B) \wedge \neg B)$
$A \to_5 B := (\neg A \wedge \neg B) \vee (\neg A \wedge B) \vee ((\neg A \vee B) \wedge A)$ (Kalmbach)

Our metalanguage consists (apart from the common parlance) of schemata from the object language as elementary metapropositions and of compound metapropositions built up by means of the following connectives: & ("and"), $\Rightarrow$ ("if, ... then") [the analogue of Kotas' (1971) "deductive scheme"] and $\Leftrightarrow$ ("iff"), with the usual "classical" meaning.

We define quantum logic QL as a schemata system containing the following axioms and rules of inference valid for all $A, B, C, \ldots \in$ Q:

Axioms
A1: $A \dashv\vdash A$
A2: $A \dashv\vdash \neg\neg A$
A3: $A \wedge B \vdash A, \quad A \wedge B \vdash B$
A4: $A \wedge \neg A \vdash B$

Rules of inference
R1: $A \vdash B \;\&\; B \vdash C \Rightarrow A \vdash C$
R2: $A \vdash B \Rightarrow \neg B \vdash \neg A$
R3: $A \vdash B \;\&\; A \vdash C \Rightarrow A \vdash B \wedge C$
R4: $A \vdash B \;\&\; \neg A \wedge B \vdash C \wedge \neg C \Rightarrow B \vdash A$

We define minimal quantum logic, MQL, as QL without R4, i.e., as a system that coincides with Goldblatt's (1974) orthologic.

We define classical logic, CL, as QL with the first premise of R4 dropped.

Whether Q refers to QL, MQL, or CL will be clear from the context.

Thus we define the relation of implication itself (denoted by $A \vdash B$ for any $A, B \in Q$) in QL, MQL, or CL in an implicit way through the foregoing axioms and rules of inference. The definition will be given yet another meaning by Theorems 3 and 4.

Following Goldblatt (1974), we consider a nonempty set of propositions Γ from Q, and say that a proposition A is Q-derivable (denoted as $\Gamma \vdash A$, which is read: Γ implies A) if there exist $B_1, \ldots, B_n$ such that $B_1 \wedge \cdots \wedge B_n \vdash A$. Since other technical definitions and theorems are not essential to the present reasoning, we refer to Goldblatt (1974), Nishimura (1980), and dalla Chiara (1977) for them.

The following schemata and rules of inference are rather trivial consequences of A3, R1, R2, and R3, and therefore valid in all the logics for all $A, B, C \in \mathrm{Q}$:

A3V: $A \vdash A \vee B, \quad B \vdash A \vee B$
R3V: $A \vdash C \;\&\; B \vdash C \Rightarrow A \vee B \vdash C$
R5: $A \vdash B \Rightarrow A \wedge C \vdash B \wedge C$
R5V: $A \vdash B \Rightarrow A \vee C \vdash B \vee C$

In order to establish the equivalence between QL and quantum logic as formulated by Goldblatt (1974) and Nishimura (1980), we prove the following result:

Theorem 1. QL is equivalent to MQL in which the orthomodularity axiom OML holds:

$$\text{OML:} \quad A \wedge (\neg A \vee (A \wedge B)) \vdash B, \qquad \text{for all } A, B \in \mathrm{Q}$$

Proof. We shall first prove that OML holds in QL. Let us construct premises for R4:

$$(\text{A1, A3V, R5}) \Rightarrow A \wedge B \vdash A \wedge (\neg A \vee (A \wedge B))$$

$$(\text{def. of } \vee, \text{A4}) \Rightarrow \neg(A \wedge B) \wedge A \wedge (\neg A \vee (A \wedge B)) \vdash C \wedge \neg C$$

The conclusion of R4 now reads $A \wedge (\neg A \vee (A \wedge B)) \vdash A \wedge B$, and by using A3 and R1, we obtain OML.

To establish the opposite, we prove that the conclusion of R4 follows from its premises in the system MQL+OML. Using the first premise, we get

$$(A \vdash B, \text{A1, R3}) \Rightarrow A \dashv\vdash A \wedge B \Rightarrow (\text{OML}) \Rightarrow B \wedge (\neg B \vee A) \vdash A$$

Using the second, we get

$$(\neg A \wedge B \vdash C \wedge \neg C, \text{A4, R2}) \Rightarrow B \vdash \neg B \vee A \Rightarrow (\text{A1, R3}) \Rightarrow B \vdash B \wedge (\neg B \vee A)$$

R1 applied to the obtained schemata gives $B \vdash A$, i.e., the conclusion of R4. ■

Alternatively, it is an easy task to construct the Lindebaum algebra, $\mathrm{Q}/\dashv\vdash$, for QL, which is an orthomodular lattice.

The following theorem serves as a bridge between CL and classical logic as usually understood.

Theorem 2. CL is equivalent to MQL in which the distributivity axiom D holds:

$$\text{D:} \qquad A \wedge (B \vee C) \dashv\vdash (A \wedge B) \vee (A \wedge C), \qquad \text{for all } A, B, C \in \mathrm{Q}$$

Proof. By using the identity $(\neg B \wedge A) \wedge \neg(A \wedge \neg B) \dashv\vdash C \wedge \neg C$ as the second premise (the only one in CL) of R4, we obtain $A \wedge (\neg A \wedge B) \vdash B$, i.e., the commensurability (see the following section) of A and B. All the other details are well known and we omit them. ■

3. THE MERGED IMPLICATIONS

The above operations of implication are the only expressions constructible by means of the other operations in QL that are reduced to the classical implication when A and B are commensurable (Hardegree, 1981a), i.e., when they satisfy the following scheme (Zeman, 1979):

$$A \wedge (\neg A \vee B) \vdash B.$$

It has been shown (Hardegree, 1981a) that the following rules hold in QL for $i = 1, \ldots, 5$, while $R(0)$ holds in CL:

$$\mathrm{R}(i): \qquad A \vdash B \Leftrightarrow C \vee \neg C \vdash A \rightarrow_i B, \qquad i = 0, 1, \ldots, 5$$

The two stated results were taken in recent literature as basic and minimal ones in quantum logic, almost as a starting point of a considerable effort to single out particular implications best suited to the deduction scheme of the logic (Hardegree, 1981a–c; Abbott, 1976; Kalmbach, 1974, 1983; Georgacarakos, 1980). In other words, it seems it has not been recognized that the "basic" and "minimal" rules $\mathrm{R}(i)$ are in fact rather strong. Namely, when we consider them not within quantum logic as usual, but in conjunction with minimal quantum logic, we realize that they turn minimal logic into quantum logic for $i = 1, \ldots, 5$, or even more into classical logic for $i = 0$. Let us prove both claims.

Theorem 3. MQL in which R(0) holds is CL, and vice versa.

Proof. The first part of the claim is nothing but a corollary of Theorem 2, and the "vice versa" part of the claim is well known. ■

Theorem 4. MQL in which at least one of $\mathrm{R}(i)$ for $i \neq 0$ holds is QL, and vice versa.

Proof. The "vice versa" part of the claim is well known.

To prove the first part of the claim, we have to prove that R4 holds in such an MQL.

$i = 1$. The first premise of R4, together with R5 and R2, gives $\neg(A \wedge B) \vdash \neg A$. For the last scheme, the second premise of R4 and R5 give $B \wedge \neg(A \wedge B) \vdash C \wedge \neg C$. After applying R2 to this scheme and using R(1) we obtain $B \vdash A$, i.e., the conclusion of R4. Hence R4 holds.

$i = 2$. Writing down R(1) for $\neg B \vdash \neg A$ and using A2 and R2, we obtain R(2).

$i = 3$. As for $i = 1$, we have $\neg(A \wedge B) \vdash \neg A$. On the other hand, the first premise of R4 and R5V give $A \vee B \vdash B$. Applying R5 and R1 to the obtained schemata, we obtain $\neg(A \wedge B) \wedge (A \vee B) \vdash \neg A \wedge B$. Applying R5 once again and using A3 and R1, we get

$$\neg(A \wedge B) \wedge \neg(A \wedge \neg B) \wedge \neg(\neg A \wedge \neg B) \vdash \neg A \wedge B$$

Now, the second premise of R4, together with R2 and R(3), gives $B \vdash A$, i.e., the conclusion of R4.

$i = 4$. The first premise of R4, A1, and R3V give $A \vee B \vdash B$, which, in combination with A3 ($\neg A \wedge B \vdash B$) and R5V, gives $A \vee (B \wedge \neg A) \vdash B$. This and (see $i = 1$) $\neg(A \wedge B) \vdash \neg A$ as in the previous case ($i = 3$) give

$$\neg(A \wedge B) \wedge \neg(A \wedge \neg B) \wedge \neg(\neg A \wedge (\neg B \vee A)) \vdash \neg A \wedge B$$

Hence (see $i = 3$) R4 holds.

$i = 5$. The first premise of R4, R5V, and the definition of disjunction give $\neg(\neg A \wedge \neg B) \vdash B$. The first premise of R4, R2, R5, and R3V give $\neg B \vee (B \wedge \neg A) \vdash \neg A$. Combining the two obtained schemata and applying R5, R3, and R1 three times, we obtain

$$\neg(\neg A \wedge \neg B) \wedge \neg(A \wedge \neg B) \wedge (\neg B \vee (B \wedge \neg A)) \vdash \neg A \wedge B$$

Hence (see $i = 3$) R4 holds. ■

Given the last theorem, we see that quantum logic is nothing but minimal logic extended so as to make "$A \to_i B$," $i = 1, \ldots, 5$, a logical truth iff "$A \vdash B$." Because of this as well as for the reason given in the following section, we propose that the implication be "defined" by R(i) $i = 1, \ldots, 5$, thus merging all five possible operations of implication "into" one relation of implication.

4. CONCLUSION

The foregoing elaboration has revealed that the property of orthomodularity can be understood as the merging of the possible operations of implication with the relation of implication. Thus, quantum logic is a logic that "works" with the help of the relation of implication, *provided* the possible operations of implication are reduced to it. On the other hand, it is a rather simple corollary of Theorem 4 combined with R1 that the metalogical modus ponens holds in quantum logic for any of the implications defined above. However, for none of the operations of implication, apart from the classical one in CL, does the proper deduction theorem hold (Kalmbach, 1974).

In our opinion, these apparently contradictory aspects of quantum logic make it feasible to consider quantum logic as a proper, though "empirical," logic, providing we keep to the following physical interpretation.

The propositions of quantum logic represent observables measurable by experiment, and their combination within the object language of the logic of measurement corresponds to possible combinations of the observables whether they are measurable or not. To handle the propositions, we must have particular metarules in the same way in which we must have a *theory* to be able to arrange experiments. The obvious disadvantage of such a situation is that the possible experimental outcomes can be divorced from the observables as such. The standard way out is to ascribe probability functions (states) to all the propositions, in other words, to merge logic with its probabilistic semantics. From our point of view (Pavičić, 1987) this boils down to a correspondence between the relation of implication—which "stands" for metarules—and an ordering relation among the probability functions.

Now, the fortunate property of any classical theory is that there is a direct correspondence, given by Theorem 3, between the relation of implication, which in turn corresponds to a particular ordering among probabilitity functions, and the unique operation of implication, for which the deduction theorem holds. The unfortunate property of any quantum theory is that there is no such unique operation in its logic, as shown by Theorem 4. However, as we have already stressed, we can *treat* the relation of implication itself in a *syntactical way*, i.e., as a part of the object language, using the schemata approach adopted above, provided we have formulated a probabilistic semantics that would justify such a move with respect to possible ascriptions of probability functions to the propositions, and the subsequent formulation of the Hilbert space description. And this is what we have done in Pavičić (1987).

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