## Contramodules

### Tomasz Brzeziński

Swansea University

Categories in Geometry Split 2007

(joint work with G Böhm and R Wisbauer)

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# Corings

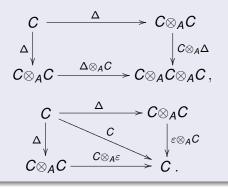
First: fix an algebra A over a commutative ring k.

### Definition

A coring (co-ring) is an A-bimodule C with A-bimodule maps

- $\Delta: C \to C \otimes_A C$  (coproduct)
- $\varepsilon : C \rightarrow A$  (counit)

such that



## Comodules

### Definition

A *right A-comodule* is a pair  $(M, \varrho)$ , where *M* is a right *A*-module, and  $\varrho : M \to M \otimes_A C$  is a right *A*-module map such that



A morphism of *C*-comodules  $(M, \varrho) \rightarrow (N, \varrho^N)$  is a right *A*-module map  $f : M \rightarrow N$  such that

$$(f \otimes_A C) \circ \varrho = \varrho^N \circ f.$$

The category is denoted by  $\mathbf{M}^{C}$ .

# Definition of contramodules

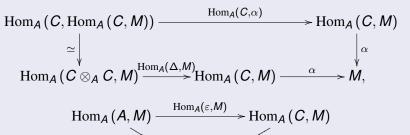
### Definition

A *C*-contramodule is a pair  $(M, \alpha)$ :

• *M* is a right *A*-module;

•  $\alpha$  : Hom<sub>A</sub> (*C*, *M*)  $\rightarrow$  *M*, is a right *A*-module map;

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### Definition

A morphism of right *C*-contramodules  $(M, \alpha_M)$ ,  $(N, \alpha_N)$  is a right *A*-module map  $f : M \to N$  rendering commutative the following diagram

The category of right *C*-contramodules is denoted by  $\mathbf{M}_{C}$ . Morphism sets (*k*-modules) are denoted by  $\operatorname{Hom}_{C}(M, N)$ .

- 1965–70: mentioned in relative homological algebra (Eilenberg-Moore), and in category theory (Vazquez Garcia, Barr);
- 2007: Positselski (arxiv:0708.3398) uses contramodules in an algebraic approach to semi-infinite cohomology (Voronov, Arkhipov).
- MathSciNet hits:
  - comodules = 797;
  - o contramodules = 3.

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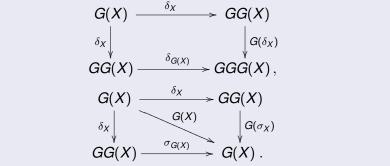
- Is there a natural explanation for the existence of contramodules and are they natural objects to consider?
- In addition to modules of a ring, are there also 'contramodules' for rings?
- What do we know about contramodules?
- Why were contramodules 'forgotten'?

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## Monads and comonads

### Definition

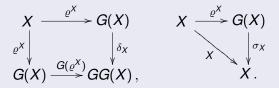
A functor  $G : \mathbf{X} \to \mathbf{X}$  is a *comonad* if there are natural transformations  $\delta : G \to GG$ ,  $\sigma : G \to id_{\mathbf{X}}$  such that, for all objects  $X \in \mathbf{X}$ ,



Monads defined dually: a functor  $F : \mathbf{X} \to \mathbf{X}$  with natural transformations  $\mu : FF \to F, \eta : \mathrm{id}_{\mathbf{X}} \to F$ .

### Definition

A *coalgebra* or *comodule* of  $(G, \delta, \sigma)$  is a pair  $(X, \varrho^X)$ , where X is an object in **X** and  $\varrho^X : X \to G(X)$  is a morphism,



A morphism of *G*-coalgebras  $(X, \varrho^X)$ ,  $(Y, \varrho^Y)$  is  $f \in \mathbf{X}(X, Y)$  such that

$$\varrho^{\mathsf{Y}}\circ f=G(f)\circ \varrho^{\mathsf{X}}.$$

Algebras of  $(F, \mu, \eta)$  defined as pairs  $(X, \varrho_X)$ , where  $\varrho_X : F(X) \to X$ .

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## The categories of algebras and coalgebras

*G*-coalgebras with their morphisms form a category  $\mathbf{X}_{G}$ .

- If **X** has colimits, coproducts, cokernels, so does **X**<sub>G</sub>.
- The forgetful functor X<sub>G</sub> → X has a right adjoint, the *free* coalgebra functor:

$$X \mapsto (G(X), \delta_X).$$

- Free coalgebras form a full subcategory of X<sub>G</sub> the Kleisli category K<sub>G</sub>.
- *F*-algebras with their morphisms form a category  $\mathbf{X}^{F}$ .
  - If **X** has limits, products, kernels, so does **X**<sup>*F*</sup>.
  - The forgetful functor X<sup>F</sup> → X has a left adjoint, the free algebra functor:

$$X \mapsto (F(X), \mu_X).$$

 Free algebras form a full subcategory of X<sup>F</sup> – the Kleisli category K<sup>F</sup>.

### Theorem (Eilenberg-Moore)

Consider an adjoint pair (L, R) of endofunctors on **X**.

- L is a comonad if and only if R is a monad.
- 2 L is a monad if and only if R is a comonad. Furthermore:

$$\mathbf{X}^{L} \equiv \mathbf{X}_{R}.$$

In the case (1),  $\mathbf{K}_L \equiv \mathbf{K}^R$ .

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#### Theorem

Let C be an A-bimodule. The following statements are equivalent:

- C is an A-coring.
- 2 The functor

$$-\otimes_{\mathcal{A}} \mathcal{C}: \mathbb{M}_{\mathcal{A}} \to \mathbb{M}_{\mathcal{A}}$$

is a comonad.

The functor

$$\operatorname{Hom}_{A}\left(\operatorname{\boldsymbol{\mathcal{C}}},-\right):\operatorname{\boldsymbol{M}}_{A}\to\operatorname{\boldsymbol{M}}_{A}$$

is a monad.

- $\mathbf{M}^{C} \equiv \text{coalgebras of comonad } (- \otimes_{A} C, \otimes_{A} \Delta, \otimes_{A} \varepsilon).$
- $\mathbf{M}_{C} \equiv \text{algebras of monad}$ (Hom<sub>A</sub>(C, -), Hom<sub>A</sub>( $\Delta$ , -), Hom<sub>A</sub>( $\varepsilon$ , -)).
- Contramodules seem to be as natural as comodules.

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## The case of algebras

#### Theorem

Let B be a k-module. The following statements are equivalent:

- B is a k-algebra.
- 2 The functor

$$-\otimes_k B: \mathbf{M}_k \to \mathbf{M}_k$$

is a monad.

The functor

$$\operatorname{Hom}_k(B,-): \mathbf{M}_k o \mathbf{M}_k$$

is a comonad.

But  $-\otimes_k B$  is the **left** adjoint of  $\operatorname{Hom}_k(B, -)$ , hence algebras of  $-\otimes_k B$  = coalgebras of  $\operatorname{Hom}_k(B, -) = \mathbf{M}_B$ 

### There are no 'contramodules' for rings.

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# 'Free' knowledge about $M_C$

- $\mathbf{M}_{C}$  has limits, products and kernels.
- $\mathbf{M}_C$  is abelian provided  $C_A$  is projective.
- The forgetful functor  $\mathbf{M}_C \to \mathbf{M}_A$  has a left adjoint, the *free contramodule* functor:

 $M \mapsto (\operatorname{Hom}_{A}(C, M), \operatorname{Hom}_{A}(\Delta, M)).$ 

- $C^* = \operatorname{Hom}_A(C, A)$  is a (free) *C*-contramodule by  $\operatorname{Hom}_A(\Delta, A)$ .
- For any contramodule  $(M, \alpha)$ ,

$$\operatorname{Hom}_{\mathcal{C}}(\mathcal{C}^*, \mathcal{M}) \simeq \mathcal{M},$$

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•  $C^*$  is a generator in  $\mathbf{M}_C$ .

## Contramodules vs modules

 $C^*$  is an algebra with the unit  $\varepsilon$  and product

$$(\xi * \xi')(c) = \sum \xi(\xi'(c_{(1)})c_{(2)}).$$

#### Theorem

• There is a faithful functor  $F : \mathbf{M}_C \to \mathbf{M}_{C^*}$  defined as follows.  $F(M, \alpha) = M$  is a right  $C^*$ -module with the action

 $\varrho_{\boldsymbol{M}}:\boldsymbol{m}\otimes\boldsymbol{\xi}\mapsto \alpha(\boldsymbol{m}\boldsymbol{\xi}(-)).$ 

For morphisms, F(f) = f.

- 2 The following statements are equivalent:
  - F is a full functor.
  - C is a finitely generated and projective right A-module.

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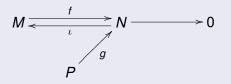
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F is an isomorphism.

# Projective contramodules

## Definition

A C-contramodule P is (C, A)-projective if any diagram



where *f*, *g* are *C*-contramodule maps and  $\iota$  is an *A*-module map, can be completed by a *C*-contramodule map  $h : P \to M$ .

#### Theorem

A C-contramodule  $(P, \alpha)$  is (C, A)-projective if and only if  $\alpha$  has a C-contramodule section. In particular, every free contramodule is (C, A)-projective.

## Definition (F Guzman)

*C* is *coseparable* if there exists an *A*-bimodule map  $\delta : C \otimes_A C \rightarrow A$  such that  $\delta \circ \Delta = \varepsilon$  and

$$(\mathcal{C} \otimes_{\mathcal{A}} \delta) \circ (\Delta \otimes_{\mathcal{A}} \mathcal{C}) = (\delta \otimes_{\mathcal{A}} \mathcal{C}) \circ (\mathcal{C} \otimes_{\mathcal{A}} \Delta).$$

#### Theorem

The following statements are equivalent:

- C is a coseparable coring.
- **2** The forgetful functor  $\mathbf{M}^C \to \mathbf{M}_A$  is separable (the unit of adjunction (Forget,  $-\otimes_A C$ ) has a natural retraction).
- The forgetful functor  $\mathbf{M}_C \to \mathbf{M}_A$  is separable (the counit of adjunction (Hom<sub>A</sub>(C, -), Forget) has a natural section).

Every contramodule of a coseparable coring is projective.

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Fix an A-coring C and a B-coring D. Aim: describe (reasonable) functors  $\mathbf{M}^D \to \mathbf{M}_C$ .

#### Theorem

Given a (C, D)-bicomodule N, and a right D-comodule M, there is a C-contramodule

 $(\operatorname{Hom}^{D}(N, M), \operatorname{Hom}^{D}(^{N}\varrho, M)).$ 

- **2** The functor  $\operatorname{Hom}^{D}(N, -) : \mathbb{M}^{D} \to \mathbb{M}_{C}$  has a left adjoint.
- ③ Any right adjoint functor  $\mathbf{M}^{D} \rightarrow \mathbf{M}_{C}$  is naturally isomorphic to Hom<sup>D</sup>(N, -) for some (C, D)-bicomodule N.

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### Definition (Positselski)

For all (C, D)-bicomodules N and right C-contramodules  $(M, \alpha)$  the *(contra)tensor product*  $M \otimes_C N$  is defined as a coequaliser

 $\operatorname{Hom}_{A}(C,M)\otimes_{A}N \xrightarrow{} M \otimes_{C}N,$ 

where the coequalised maps are  $f \otimes_A n \mapsto (f \otimes_A N) \circ {}^N_{\mathcal{Q}}(n)$  and  $\alpha \otimes_A N$ . Here  ${}^N_{\mathcal{Q}}$  is the left *C*-coaction on *N*.

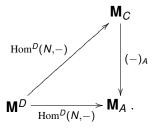
 $M \otimes_C N$  is a coequaliser of right *D*-comodule maps, hence  $M \otimes_C N$  is a *D*-comodule. There is a functor  $- \otimes_C N : \mathbf{M}_C \to \mathbf{M}^D$ .

#### Theorem (Positselski)

The functor  $-\otimes_C N : \mathbf{M}_C \to \mathbf{M}^D$  is the left adjoint of  $\operatorname{Hom}^D(N, -)$ .

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A (C, D)-bicomodule N determines a commutative diagram of right adjoint functors



There is a corresponding monad morphism

 $\operatorname{can}^{N}:\operatorname{Hom}_{\mathcal{A}}(\mathcal{C},-)\to\operatorname{Hom}^{D}(\mathcal{N},-\otimes_{\mathcal{A}}\mathcal{N}),\quad\operatorname{can}_{Q}^{N}(f)=(f\otimes_{\mathcal{A}}\mathcal{N})\circ^{N}\varrho.$ 

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#### Theorem

The functor  $- \otimes_C N : \mathbf{M}_C \to \mathbf{M}^D$  is fully faithful if and only if the following assertions hold.

- (i) The natural transformation  $can^N$  is an isomorphism.
- (ii) For all contramodules (M, α), the functor Hom<sup>D</sup>(N, −) : M<sup>D</sup> → M<sub>A</sub> preserves the coequaliser defining M⊗<sub>C</sub>N.

### Corollary

If  $\operatorname{can}^N$  is a natural isomorphism and  $\operatorname{Hom}^D(N, -) : \mathbf{M}^D \to \mathbf{M}_A$  is a right exact functor, then  $- \otimes_C N : \mathbf{M}_C \to \mathbf{M}^D$  is fully faithful and *C* is a projective right A-module.

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### Theorem (Beck's theorem)

The categories  $\mathbf{M}_C$  and  $\mathbf{M}^D$  are equivalent if and only if there exists a (C, D)-bicomodule N such that

- (i) The natural transformation  $can^N$  is an isomorphism.
- (ii) The functor  $\operatorname{Hom}^D(N, -) : \mathbf{M}^D \to \mathbf{M}_A$  reflects isos.
- (iii) The functor  $\operatorname{Hom}^{D}(N, -) : \mathbb{M}^{D} \to \mathbb{M}_{A}$  preserves reflexive  $\operatorname{Hom}^{D}(N, -)$ -contractible coequalisers.

### Corollary

## For a (C, D)-bicomodule N, TFAE:

(i)  $\operatorname{Hom}^{D}(N, -) : \mathbf{M}^{D} \to \mathbf{M}_{C}$  is an equivalence and C is a projective right A-module.

(ii)  $\operatorname{can}^N$  is a natural iso, N is a generator in  $\mathbb{M}^D$  and  $\operatorname{Hom}^D(N, -) : \mathbb{M}^D \to \mathbb{M}_A$  is right exact.