RENORMALIZATION IN THE COULOMB GAUGE

RAB, June 2006
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OUTLINE

INTRODUCTION

1. HAMILTONIAN FORMALISM
2. TWO-POINT FUNCTIONS
3. PROPAGATORS
4. ZWANZIGER'S LIMIT
5. THREE-POINT FUNCTIONS
6. SLAVNOV-TAYLOR IDENTITIES
7. COUNTER-TERMS IN X-SPACE ?

CONCLUSION

THE COULOMB GAUGE $\quad \partial_i A_i = 0$

ADVANTAGES

a) NO GHOST STATES

GHOST PROPAGATOR $\frac{1}{k^2}$ HAS NO POLES

b) PHYSICAL GAUGE – MANIFESTLY UNITARY

DISADVANTAGES

i) HAMILTONIAN IS NON-LOCAL & NON-POLYNOMIAL

ii) ENERGY DIVERGENCES

$$\int \frac{d^3P}{(2\pi)^3} \int \frac{dp_0}{(2\pi)} \frac{p_0^2}{p_0^2 - P^2 + i\eta} \times \frac{1}{(P-K)^2}$$

CAUCHY IN p_0 PLANE – DIVERGES!

NO REGULARIZATION PROCEDURE FOR DIV. IN p_0

P. DOUST & J.C. TAYLOR, PHYS. LETT. 197 (1987) 232

P. DOUST, ANN. OF PHYS. 177 (1987) 169

SYSTEMATIC CANCELLATIONS – NO GENERAL PROOF THAT CONTROLS SUCH DIVERGENCES

G. LEIBBRANDT, NUCL. PHYS. B 575 (2000) 359

SPLIT DIMENSIONAL REGULARIZATION

iii) A COMPLETE DISCUSSION OF RENORMALIZATION IN THE COULOMB GAUGE HAS NOT YET BEEN GIVEN

HAMILTONIAN FORMALISM

GENERATING FUNCTIONAL

$$Z(j,J) = \int d[F]\, d[A]\, [\, j^\mu A_\mu + J^{\mu\nu} F_{\mu\nu}]\, \exp\, [-i \int d^4x\, L\,]$$

J, j – SOURCES (USED TO GENERATE GREEN'S FUNCTIONS)

L – LAGRANGIAN

$F_{\mu\nu}$, A – FIELDS

$$L = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \tfrac{1}{2} F^{a\mu\nu} f^a_{\mu\nu}$$

$$f^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

$\mu = 0, 1, 2, 3$

$i = 1, 2, 3$

COLOUR INDICES a, b, c

METRIC $g_{00} = +1$, $g_{11} = -1$ etc.

LAGRANGIAN FORMALISM SET $J = 0$

PERFORM INTEGRATION $d[F]$

GET LAGRANGIAN $-\frac{1}{4} f^2$

HAMILTONIAN FORMALISM KEEP SOME J_{0i}

WRITE $\mathcal{L}$ AS

$$\mathcal{L} = -\frac{1}{4}(F_{ij}^a)^2 + \frac{1}{2}(F_{0i}^a)^2 + \frac{1}{2} F_{ij}^a f_{ij}^a - F_{0i}^a f_{0i}^a$$

SET $J_{ij} = 0$

PERFORM INTEGRAL OVER F_{ij} (NO TIME DERIVATIVES INVOLVED)

EFFECT

$$\mathcal{L} = -\frac{1}{4}(f_{ij}^a) - \frac{1}{2}(F_{0i}^a)^2 + F_{0i}^a f_{0i}^a$$

WHERE

$$F_{0i}^a f_{0i}^a = F_{0i}^a [\partial_0 A_i^a - \partial_i A_0^a - g f^{abc} A_0^b A_i^c]$$

COMPARE WITH PARTICLE THEORY

$$\mathcal{L} = p\dot{q} - H$$

F_{0i} – MOMENTUM CONJUGATE TO A_i

ADD THE COULOMB GAUGE FIXING TERM

$$\frac{1}{2\alpha}(\partial_i A_i)^2$$

PROPAGATORS

WRITE $F_{0i} = E_i$

$$-\frac{1}{4}[\partial_i A_j - \partial_j A_i]^2 + \frac{1}{2\alpha}(\partial_i A_i)^2 - \frac{1}{2}E_i^2 + E_i \dot{A}_i - E_i \partial_i A_0$$

ESSENTIAL

HAMILTONIAN FORMALISM IS FIRST ORDER IN TIME DERIVATIVES

7×7 MATRICES ACTING ON $A_1, A_2, A_3; A_0; E_1, E_2, E_3$:

	A_j	A_0	E_n
A_i	$-k^2(T_{ij} + L_{ij}/\alpha)$	0	$-ik_0\delta_{in}$
A_0	0	0	$-iK_n$
E_m	$ik_0\delta_{mj}$	$+iK_m$	$-\delta_{mn}$

$$T_{ij} \equiv \delta_{ij} - L_{ij} \quad , \qquad L_{ij} = \frac{K_i K_j}{K^2}$$

PROPAGATOR IS THE INVERSE OF THIS MATRIX

	A_j	A_0	E_n
A_i	$+T_{ij}/k^2 - \alpha L_{ij}/K^2$	$\alpha k_0 K_i/K^4$	$-ik_0 T_{in}/K^2$
A_0	$\alpha k_0 K_j/K^4$	$1/K^2 + \alpha k_0^2/K^4$	$-iK_n/K^2$
E_m	$+ik_0 T_{mj}/k^2$	iK_m/K^2	$+T_{mn}K^2/k^2$

$$k^2 = k_0^2 - K^2$$

COULOMB GAUGE SET $\alpha = 0$

i) OFF DIAGONAL TERMS : A, E, A_0 MIX

ii) EE PART HAS K^2 IN THE NUMERATOR, NOT k_0^2 (AS IN ORDINARY LAGRANGE THEORY)

D. ZWANZIGER, NUCL. PHYS. B 518 (1998) 237
— JUST UV POLE PARTS TO ORDER g^2

FEYNMAN RULES

· · · · · · · · · · · A_0 INSTANTANEOUS COULOMB PROPAGATOR

— — — — — — — A_i TRANSVERSE PROPAGATOR

———————— E_i MOMENTUM CONJUGATE TO A_i

i — — — A_i — — — j $$\frac{1}{k^2+i\epsilon}\left[\delta_{ij}-\frac{k_i k_j}{\mathbf{k}^2}\right]$$

0 · · · · · A_0 · · · · · 0 $$\frac{1}{\mathbf{k}^2}$$

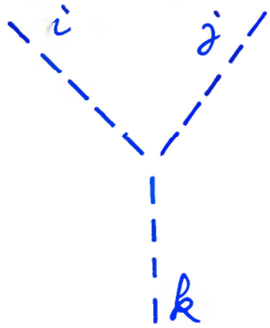
m ———————— n $$\frac{\mathbf{k}^2}{k^2+i\epsilon}\left[\delta_{mn}-\frac{k_m k_n}{\mathbf{k}^2}\right]$$

WHERE $k^2 = k_0^2 - \mathbf{k}^2$

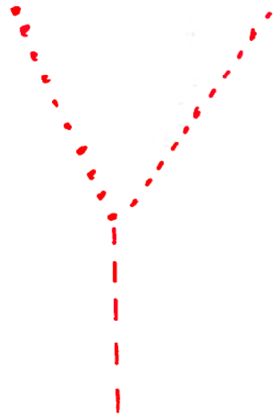
A_i — — — — ———— E_j $$-\frac{ik_0}{k^2+i\epsilon}\left[\delta_{ij}-\frac{k_i k_j}{\mathbf{k}^2}\right]$$

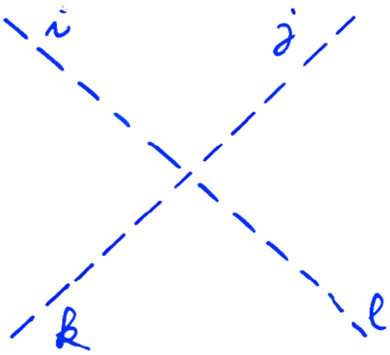
A_0 · · · · · · ———— E_i $$-\frac{ik_i}{\mathbf{k}^2}$$

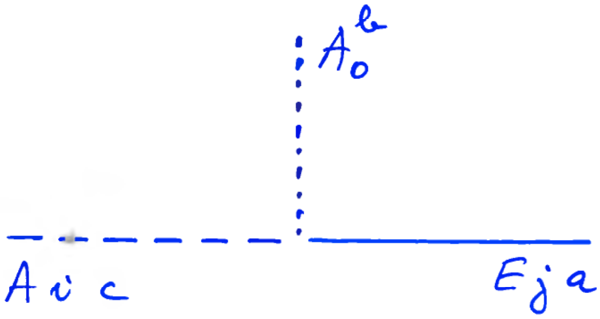
VERTICES



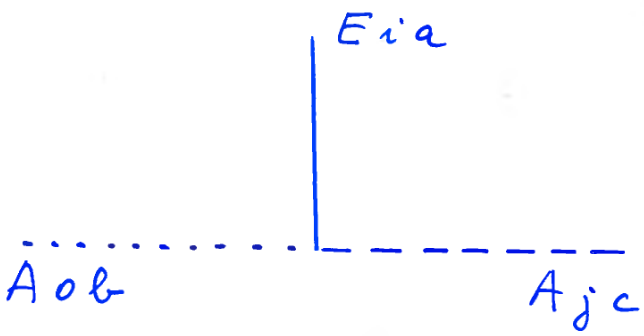
NOTE, NO SUCH VERTEX!







$$g f^{abc} \delta_{ij}$$



$$g f^{abc} \delta_{ij}$$

REMEMBER GHOSTS

$$\frac{1}{\alpha}(\partial_i A_i)^2 \longrightarrow c^* \partial_i D_i(A) c$$

$D(A)$ – COVARIANT DERIVATIVE

CLOSED LOOPS WITH MINUS SIGN

CANCEL EACH OTHER

FORGET GHOSTS AS LONG AS WE OMIT COULOMB CLOSED LOOPS

MOTIVATION

$$\sigma_{Coul} \equiv \lim_{R\to\infty} V(R)/R \qquad \text{STRING TENSION}$$

D. ZWANZIGER, NUCL. PHYS. B 518 (1998) 237

POTENTIAL $$V(K) = \lim_{k_0\to\infty} g^2 D^{A_0 A_0}(k_0, K)$$

$k_0 \to \infty$

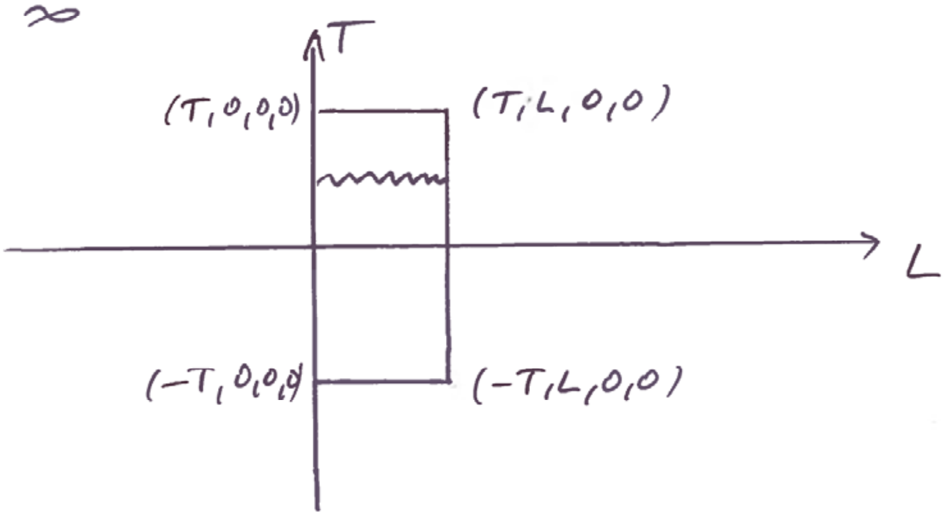
$\delta(x_0)$ – INSTANTANEOUS IN POSITION SPACE

J.C. TAYLOR $\quad k_0 \to 0 \quad$ QUARK–ANTIQUARK POTENTIAL

WILSON LOOP

$(T,0,0,0)$, $(-T,0,0,0)$, $(T,L,0,0)$, $(-T,L,0,0)$

$T \to \infty$



$$D_{\mu\nu}(k_0,K) \longrightarrow D_{00}(k_0,K)$$

$$W = \int d^4k \int_{-T}^{T} dt \int_{-T}^{T} dt' \, e^{i(t-t')k_0} \, e^{iL\cdot K} D_{00}(k_0,K)$$

$$= \int d^4k \left(\frac{2\sin Tk_0}{k_0}\right)^2 e^{iL\cdot K} D_{00}(k_0,K)$$

REPREZENTATION OF δ

$$\delta(k_0) = \frac{1}{\pi} \lim_{T\to\infty} \frac{\sin Tk_0}{k_0}$$

i.e. $\quad \frac{2\sin Tk_0}{k_0} \times 2\pi\delta(k_0)$

$$W = 4\pi T \int d^3K \, e^{iK\cdot L} D_{00}(k_0 \to 0, K)$$

ZERO'th ORDER $\to$ COULOMB POTENTIAL $\quad \frac{1}{L}$

A_0A_0 TWO-POINT FUNCTION

$$G^1_{00} = g^2 C_G \delta_{ab} \int d^4p \, \frac{p_0}{p^2+i\eta} \left(\delta_{ij} - \frac{p_i p_j}{\underset{\sim}{p}^2}\right) \times \frac{(k-p)_0}{(k-p)^2+i\eta} \left(\delta_{ij} - \frac{(k-p)_i (k-p)_j}{(\underset{\sim}{k}-\underset{\sim}{p})^2}\right)$$

$$= g^2 C_G \delta_{ab} \int d^{4-\epsilon}p \, \frac{p_0}{p^2+i\eta} \, \frac{(k-p)_0}{(k-p)^2+i\eta} \left[1-\epsilon + \frac{(\underset{\sim}{p}\cdot(\underset{\sim}{k}-\underset{\sim}{p}))^2}{\underset{\sim}{p}^2 (\underset{\sim}{k}-\underset{\sim}{p})^2}\right]$$

$$p_0 (k-p)_0 = p_0 k_0 - \underset{\sim}{p}^2 - \underbrace{p^2}_{\text{4-vector}}$$

HOW TO DEAL WITH INTEGRALS CONTAINING INSTANTANEOUS COULOMB PROPAGATORS?

BASIC INTEGRAL

$$I = \int d^4 p \, \frac{p_0}{p^2 + i\eta} \, \frac{1}{(k-p)^2 + i\eta}$$

SCHWINGER REPRESENTATION

$$\frac{1}{k^2 - m^2 + i\eta} = -i \int_0^\infty d\alpha \, e^{i\alpha(k^2 - m^2 + i\eta)}$$

$$I = -\int_{-\infty}^{\infty} dp_0 \, p_0 \int d^{3-\epsilon} p \int_0^\infty d\alpha \int_0^\infty d\beta$$

$$\times \, e^{i\{p^2(\alpha+\beta) - 2pk\alpha + \alpha k^2 + i\epsilon(\alpha+\beta)\}}$$

$$I = -\int_0^\infty d\alpha \int_0^\infty d\beta \int_{-\infty}^{\infty} dp_0 \, p_0 \, e^{i\{p_0^2(\alpha+\beta) - 2p_0 k_0 \alpha\}}$$

$$\times \int d^{3-\epsilon} p \, e^{-i\{\underline{p}^2(\alpha+\beta) - 2\underline{p}\cdot\underline{k}\,\alpha\}} \times e^{i\alpha k^2 - \epsilon(\alpha+\beta)}$$

$$\int_{-\infty}^{\infty} e^{i(at^2 + 2bt)} dt = \frac{1+i}{\sqrt{2}} e^{-\frac{ib^2}{a}} \int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{1+i}{\sqrt{2}} \sqrt{\frac{\pi}{a}} \, e^{-\frac{ib^2}{a}}$$

$(a > 0)$

APPLY $\frac{\partial}{\partial b}$ ON BOTH SIDES

$$\int_{-\infty}^{\infty} e^{i(at^2 + 2bt)} t \, dt = -\frac{b}{a} \sqrt{\frac{\pi}{a}} \, \frac{1+i}{\sqrt{2}} \, e^{-\frac{ib^2}{a}}$$

$$I = k_0 \frac{1+i}{\sqrt{2}} \sqrt{\pi} \int_0^\infty d\alpha \int_0^\infty d\beta \; e^{i\alpha k^2 - \epsilon(\alpha+\beta)}$$

$$\times \frac{\alpha}{(\alpha+\beta)^{3/2}} \; e^{-i \frac{k_0^2 \alpha^2}{\alpha+\beta}} \times \int d^{3-\epsilon} p \; e^{-i\{p^2(\alpha+\beta) - 2 p\cdot k \, \alpha\}}$$

PERFORM PARAMETER INTEGRATION TO RETURN TO NORMAL FEYNMAN-LIKE PROPAGATORS IN $3-\epsilon$ DIM.

$$\alpha = \xi\lambda \quad , \quad \beta = (1-\xi)\lambda$$

$$\left(\frac{\partial\alpha, \partial\beta}{\partial\xi, \partial\lambda}\right) = \lambda \qquad\qquad \begin{array}{l} 0 < \xi < 1 \\ 0 < \lambda < \infty \end{array}$$

$$I = k_0 \, e^{\frac{i\pi}{4}} \sqrt{\pi} \int_0^\infty \lambda \, d\lambda \int_0^1 d\xi \, \frac{\xi\lambda}{\lambda^{3/2}} \; e^{i\xi\lambda k^2 - \epsilon\lambda}$$

$$\times e^{-\frac{i k_0^2 \lambda^2 \xi^2}{\lambda}} \times \int d^{3-\epsilon} p \; e^{-i\{p^2\lambda - 2 p\cdot k \, \xi\lambda\}}$$

$$I = k_0 \, e^{\frac{i\pi}{4}} \sqrt{\pi} \int_0^1 \xi \, d\xi \int_0^\infty d\lambda \, \lambda^{\frac{1}{2}}$$

$$\times e^{-\lambda\{\epsilon + i[p^2 - 2 p\cdot k \, \xi - \xi k^2 + \xi^2 k_0^2]\}}$$

NOW PERFORM λ-INT. WITH THE FORMULA

$$\int_0^\infty x^{\nu-1} e^{-\mu x} dx = \frac{1}{\mu^\nu} \Gamma(\nu)$$

$$[\,\mathrm{Re}\,\mu > 0 \;, \mathrm{Re}\,\nu > 0\,]$$

$$I = k_0\, e^{-\frac{i\pi}{2}} \sqrt{\pi}\, \Gamma\left(\tfrac{3}{2}\right) \int_0^1 \xi\, d\xi$$

$$\times \int d^{3-\epsilon} p \,\frac{1}{\left\{ \underset{\sim}{p}^2 - 2\underset{\sim}{p}\cdot\underset{\sim}{k}\,\xi - \xi k^2 + \xi^2 k_0^2 - i\epsilon \right\}^{3/2}}$$

EXAMPLE OF COULOMB GAUGE INTEGRALS

$$T = \int d^4 p \,\frac{p_0 k_0}{(p^2 + i\eta)[(k-p)^2 + i\eta]} \times \frac{2\underset{\sim}{p}\cdot\underset{\sim}{k}}{\underset{\sim}{p}^2}$$

$$T = -i k_0^2 \sqrt{\pi}\, \Gamma\left(\tfrac{3}{2}\right) \int_0^1 \xi\, d\xi \int d^{3-\epsilon} p \,\frac{2\underset{\sim}{p}\cdot\underset{\sim}{k}}{\left\{ \qquad \right\}^{3/2} \underset{\sim}{p}^2}$$

COMBINE DENOMINATORS

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \,\frac{x^{\alpha-1}(1-x)^{\beta-1}}{\left\{ ax + b(1-x) \right\}^{\alpha+\beta}}$$

$$T = -i k_0^2 \sqrt{\pi}\, \Gamma\left(\tfrac{5}{2}\right) \int_0^1 \xi\, d\xi \int_0^1 dx\, x^{\frac{1}{2}}$$

$$\times \int d^{3-\epsilon} p \,\frac{2\underset{\sim}{p}\cdot\underset{\sim}{k}}{\left\{ \underset{\sim}{p}^2 - 2\underset{\sim}{p}\cdot\underset{\sim}{k}\, x\xi - x\xi k^2 + x\xi^2 k_0^2 - i\eta x \right\}^{5/2}}$$

NOW $d^{3-\epsilon} p$ INT. IS EASY

$$T = -2i\pi^{\frac{4-\epsilon}{2}} k_0^2 \underset{\sim}{k}^2 \Gamma\left(\tfrac{2+\epsilon}{2}\right) \int_0^1 d\xi\, \xi^{1-\frac{\epsilon}{2}} \int_0^1 dx\, x^{\frac{1-\epsilon}{2}}$$

$$\times \left\{ -k^2 - \frac{i\eta}{\xi} + \xi\left(k_0^2 - x\underset{\sim}{k}^2\right)\right\}^{-\frac{2+\epsilon}{2}}$$

4 vector

$$T \sim \int_0^1 d\xi (1-\xi) \int_0^1 dx (1-x)^{\frac{1}{2}} \left\{ -\xi k^2 - i\eta + x(1-\xi)\underset{\sim}{k}^2 \right\}^{-1}$$

MAIN PROBLEM NEAR $x \sim 1$, $\xi \sim 1$

FOR $k_0^2 > \underset{\sim}{k}^2$ $\{\ \}$ VANISHES AT $x = \frac{\xi k^2}{\underset{\sim}{k}^2}$

EVALUATE T IN THE REGION $k_0^2 < \underset{\sim}{k}^2$ WHERE IT IS WELL DEFINED AND ANALITICALLY CONTINUE TO $k_0^2 > \underset{\sim}{k}^2$.

OMIT THE CONSTANT $C = -2i\pi^{\frac{4-\epsilon}{2}} k_0^2 \underset{\sim}{k}^2 \Gamma\left(\frac{2+\epsilon}{2}\right)$

SET $\epsilon = 0$ (AS T IS IR FINITE)

$$T = \int_0^1 d\xi\, \xi \int_0^1 dx\, x^{\frac{1}{2}} \times \left\{ -k^2 - \frac{i\eta}{\xi} + \xi\left(k_0^2 - x\underset{\sim}{k}^2\right)\right\}^{-1}$$

AS $i\eta$ SHOWS ONLY THE DIRECTION OF ANALYTIC CONTINUATION, WE CAN WRITE

(FOR SIMPLICITY OF WRITING, SET

$\xi = y$, $\underset{\sim}{k}^2 = K^2$)

$$T = \int_0^1 dy\, y \int_0^1 dx \frac{x^{\frac{1}{2}}}{\left\{ -k^2 - i\eta + y\left(k_0^2 - x(K^2 - i\eta)\right)\right\}}$$

AFTER dy INT.

$$T = \int_0^1 dx \, \frac{x^{\frac{1}{2}}}{k_0^2 - x(k^2 - i\eta)}$$

$$+ (k^2 + i\eta) \int_0^1 dx \, \frac{x^{\frac{1}{2}}}{[k_0^2 - x(k^2 - i\eta)]^2} \ln \frac{k^2(1-x)}{(-k^2 - i\eta)}$$

FIRST INT. AFTER SUBST. $x = y^2$

$$A = \int_0^1 dx \, \frac{x^{\frac{1}{2}}}{k_0^2 - x(k^2 - i\eta)} = -\frac{2}{k^2} + \frac{k_0}{k^3} \ln \frac{k_0 + k - i\eta}{k_0 - k + i\eta}$$

$$B = \int_0^1 dx \, \frac{x^{\frac{1}{2}}}{(k_0^2 - x(k^2 - i\eta))^2} = -\frac{\partial}{\partial k_0^2} A$$

$$C = \int_0^1 dx \, \frac{x^{\frac{1}{2}} \ln(1-x)}{(k_0^2 - x(k^2 - i\eta))^2}$$

TRICK – INTRODUCE A PARAMETER t

$$C(t) = \int_0^1 dx \, \frac{x^{\frac{1}{2}}}{[k_0^2 - x(k^2 - i\eta)]^2} \ln(1 - tx)$$

$$C(1) = C$$
$$C(0) = 0$$

$$\frac{\partial}{\partial t} C(t) = \int_0^1 dx \, \frac{x^{\frac{1}{2}}}{[k_0^2 - x(k^2 - i\eta)]^2} \times \frac{(-x)}{1 - tx}$$

INTEGRATE BY PARTS

$$\frac{\partial}{\partial t} C(t) = -\frac{1}{K^2} \times \frac{1}{k_0^2 - x(K^2 - i\eta)} \times \frac{x^{\frac{3}{2}}}{1-tx} \Bigg|_{x=0}^{x=1}$$

$$+ \frac{1}{K^2} \int_0^1 dx \frac{1}{k_0^2 - x(K^2 - i\eta)} \left\{ \frac{3 x^{\frac{1}{2}}}{2(1-tx)} + \frac{x^{\frac{3}{2}}}{(1-tx)^2} t \right\}$$

$$\frac{\partial}{\partial t} C(t) = -\frac{1}{K^2} \frac{1}{k_0^2 - K^2 + i\eta} \frac{1}{1-t}$$

$$+ \frac{1}{2K^2} \int_0^1 dx \frac{1}{k_0^2 - x(K^2 - i\eta)} \times \frac{x^{\frac{1}{2}}}{1-tx}$$

$$+ \frac{1}{K^2} \int_0^1 dx \frac{1}{k_0^2 - x(K^2 - i\eta)} \times \frac{x^{\frac{1}{2}}}{(1-tx)^2}$$

NOW INTEGRATE IN t

$$C = \int_0^{1-\delta} C(t)\, dt$$

$$C = \frac{1}{K^2} \frac{1}{k_0^2 - K^2 + i\eta} \ln \delta$$

$$- \frac{1}{2K^2} \int_0^1 dx \frac{x^{-\frac{1}{2}}}{k_0^2 - x(K^2 - i\eta)} \ln(1-x)$$

$$+ \frac{1}{K^2} \int_0^1 dx \frac{x^{-\frac{1}{2}}}{k_0^2 - x(K^2 - i\eta)} \times \frac{1}{1-(1-\delta)x}$$

$$- \frac{1}{K^2} \int_0^1 dx \frac{x^{-\frac{1}{2}}}{k_0^2 - x(K^2 - i\eta)}$$

$$\frac{1}{k_0^2 - x(K^2 - i\eta)} \times \frac{1}{1-(1-\delta)x} = \frac{1}{k_0^2 - K^2 + i\eta}$$

$$\times \left\{ \frac{1}{1-(1-\delta)x} - \frac{K^2}{k_0^2 - x(K^2 - i\eta)} \right\}$$

$$C = \frac{1}{K^2} \times \frac{1}{k_0^2 - K^2 + i\eta} \ln\delta$$

$$- \frac{1}{2K^2} \int_0^1 dx \, \frac{x^{-\frac{1}{2}}}{k_0^2 - x(K^2 - i\eta)} \ln(1-x)$$

$$- \frac{1}{K^2} \int_0^1 dx \, \frac{x^{-\frac{1}{2}}}{k_0^2 - x(K^2 - i\eta)}$$

$$+ \frac{1}{K^2} \frac{1}{k_0^2 - K^2 + i\eta} \left\{ \int_0^1 dx \, \frac{x^{-\frac{1}{2}}}{1-(1-\delta)x} - \int_0^1 dx \, \frac{K^2 x^{-\frac{1}{2}}}{k_0^2 - x(K^2 - i\eta)} \right\}$$

THE INTEGRAL WHICH INSURES CANCELLATION OF $\ln\delta$

$$M = \int_0^1 dx \, \frac{x^{-\frac{1}{2}}}{1-(1-\delta)x}$$

$$x = y^2$$

$$M = 2\ln 2 - \ln\delta$$

$$C = \frac{1}{K^2} \frac{1}{k_0^2 - K^2 + i\eta} \times 2\ln 2 - \frac{1}{2K^2} \int_0^1 dx \, \frac{x^{-\frac{1}{2}}}{k_0^2 - x(K^2 - i\eta)} \ln(1-x)$$

$$- \frac{1}{K^2} \int_0^1 dx \, \frac{x^{-\frac{1}{2}}}{k_0^2 - x(K^2 - i\eta)} - \frac{1}{k_0^2 - K^2 + i\eta} \int_0^1 dx \, \frac{x^{-\frac{1}{2}}}{k_0^2 - x(K^2 - i\eta)}$$

SUBST. $x = y^2$ $\qquad k_0 = |k_0|$, $K = |\underline{k}|$

$$N = \int_0^1 dx \, \frac{x^{-\frac{1}{2}}}{k_0^2 - x(K^2 - i\eta)} = \frac{1}{k_0 K} \ln \frac{k_0 + K - i\eta}{k_0 - K + i\eta}$$

$$D = \int_0^1 dx \, \frac{x^{-\frac{1}{2}}}{k_0^2 - x(K^2 - i\eta)} \ln(1-x)$$

$$= \frac{1}{k_0} \int_0^1 dy \left\{ \ln(1-y) + \ln(1+y) \right\} \left\{ \frac{1}{k_0 - y(K - i\eta)} + \frac{1}{k_0 + y(K - i\eta)} \right\}$$

$\longrightarrow$ SPENCE FUNCTIONS

FINAL RESULT

$$T = \frac{2}{K^2}(\ln 2 - 1) + \left[\frac{1}{K^2} - \frac{k^2}{2k_0 K^3} \ln \frac{k_0 + K - i\eta}{k_0 - K + i\eta} \right] \times \ln \frac{K^2}{(-k^2 - i\eta)}$$
$$- \frac{k^2}{2K^2} D$$

IN THE REGION $k_0 > K$

$$D = \frac{1}{k_0 K} \left\{ Li_2\left(\frac{k_0 - K + i\eta}{k_0 + K - i\eta} \right) - Li_2\left(\frac{k_0 + K - i\eta}{k_0 - K + i\eta} \right) \right.$$
$$\left. + \ln \frac{k_0 + K - i\eta}{k_0 - K + i\eta} \times \ln \frac{k^2 + i\eta}{K^2} - i\pi \ln \frac{k_0 + K - i\eta}{k_0 - K + i\eta} \right\}$$

SPENCE FUNCTION

$$Li_2(x) = - \int_0^x \frac{\ln(1-z)}{z} \, dz$$

IN THE REGION $K > k_0$

$$D = \frac{1}{k_0 K}\left\{ Li_2\left(\frac{K-k_0-i\eta}{K+k_0-i\eta}\right) - Li_2\left(\frac{K+k_0-i\eta}{K-k_0-i\eta}\right)\right.$$

$$+ \ln\frac{K+k_0-i\eta}{K-k_0-i\eta} \times \ln\left(\frac{-k^2-i\eta}{k_0^2}\right) + i\pi \ln\frac{K+k_0-i\eta}{K-k_0-i\eta}\Bigg\}$$

$$-\frac{2}{k_0 K}\left[Li_2\left(-\frac{k_0}{K-i\eta}\right) - Li_2\left(\frac{k_0}{K-i\eta}\right)\right]$$

$$+\frac{i\pi}{k_0 K}\ln\frac{K^2}{(-k^2-i\eta)}$$

TWO-POINT FUNCTIONS

$$\Gamma_a^{A_0 A_0} = c\left\{\Gamma\left(\frac{\epsilon}{2}\right)\left(\frac{-k^2-i\eta}{\mu^2}\right)^{-\frac{\epsilon}{2}} \times \left(\frac{1}{2}k_0^2 + \frac{5}{6}K^2 + \frac{\epsilon}{12}k^2 + \frac{\epsilon}{6}k_0^2 + \frac{17}{18}\epsilon K^2\right)\right.$$

$$-2^{-\epsilon}\left(\frac{5}{3} + \frac{28}{9}\epsilon\right)\Gamma\left(\frac{\epsilon}{2}\right)K^2\left(\frac{K^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}$$

$$+\frac{1}{2}k^4 \times D$$

$$+\frac{k^4}{2k_0 K}\ln\frac{k_0+K-i\eta}{k_0-K+i\eta} \times \ln\frac{K^2}{(-k^2-i\eta)}$$

$$\left. - k_0^2 \ln\frac{K^2}{(-k^2-i\eta)} - 2(\ln 2 - 1)k_0^2\right\}$$

$$c = \frac{i g^2}{16\pi^2} C_G \delta_{ab}$$

$$\Gamma_F^{A_0A_0} = c\Big\{\Gamma\left(\tfrac{\epsilon}{2}\right)\left(\frac{-k^2-i\eta}{\mu^2}\right)^{-\frac{\epsilon}{2}} \times \left(\tfrac{1}{3}K^2 - \tfrac{1}{2}k^2 - \tfrac{\epsilon}{4}k^2 + \tfrac{11}{18}\epsilon K^2\right)$$

$$-\tfrac{1}{3}\Gamma\left(\tfrac{\epsilon}{2}\right)K^2\left(\frac{K^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} - K^2\left(\frac{10}{9} - \frac{2\ln 2}{3}\right)$$

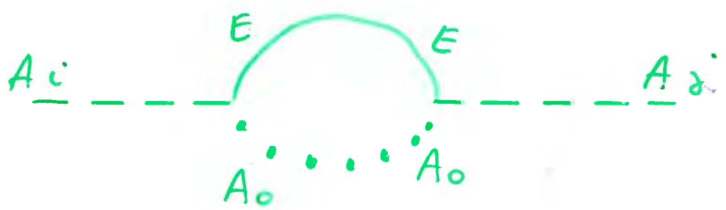
$$+ k^2 k_0^2 \times D$$

$$+ \frac{k_0 k^2}{K}\ln\frac{k_0+K-i\eta}{k_0-K+i\eta} \times \ln\frac{K^2}{(-k^2-i\eta)}$$

$$-(2k^2+K^2)\ln\frac{K^2}{(-k^2-i\eta)} - 2(2k^2+K^2)(\ln 2 - 1)\Big\}$$

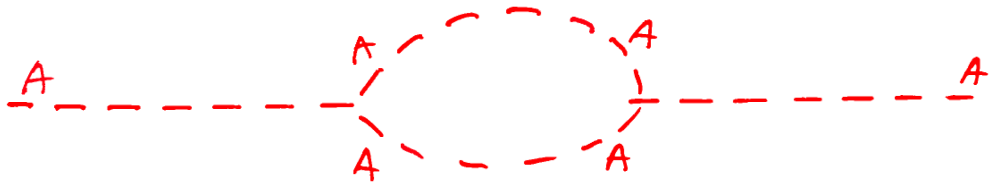
THIS IS NOT YET THE PROPAGATOR!
NEED ALL TWO-POINT FUNCTIONS
7×7 MATRIX

A_iA_j TWO-POINT FUNCTION



$$\Gamma_1^{A_iA_j} = c\,\Gamma\left(\tfrac{\epsilon}{2}\right)\left(\frac{K^2}{\mu^2}\right)^{-\frac{\epsilon}{2}} \times 2^{-\epsilon}\left(1 + \frac{77}{30}\epsilon\right)$$

$$\times\left\{\frac{12}{15}K^2\delta_{ij} - \frac{16}{15}K_iK_j + \frac{4}{15}\epsilon K_iK_j\right\}$$



$$\Gamma_2^{A_iA_j} = c\left(M K_i K_j + N K^2 \delta_{ij}\right)$$

$$M = \frac{31}{15}\Gamma\left(\frac{\epsilon}{2}\right) - \frac{4}{3}\ln\left(\frac{-k^2-i\eta}{\mu^2}\right) + \frac{3}{10}\ln\frac{K^2}{\mu^2}$$

$$+\frac{1}{4}\left[-\frac{K^2}{k_0^2} + 18\frac{k_0^2k^2}{K^2} + 9\frac{k_0^2k^4}{K^4} + k_0^2\right] \times D$$

$$+\frac{1}{4}\left[-\frac{K^3}{k_0^3} + 18\frac{k_0k^2}{K^3} + 9\frac{k_0k^4}{K^5} + \frac{k_0}{K}\right]\ln\frac{k_0+K-i\eta}{k_0-K+i\eta} \times \ln\frac{K^2}{(-k^2-i\eta)}$$

$$-\frac{1}{2K^2}\left[3k_0^2 - 5K^2 + 9\frac{k_0^4}{K^2} - \frac{K^4}{k_0^2}\right]\ln\frac{K^2}{(-k^2-i\eta)}$$

$$-\frac{\ln 2}{K^2}\left[6\frac{k_0^4}{K^2} + 9k^2 - \frac{1}{5}K^2 + 3\frac{k^4}{K^2} - \frac{K^4}{k_0^2}\right]$$

$$+22\frac{k_0^2}{K^2} - 16 - \frac{1}{9} - \frac{8}{15} + 4 \times \frac{77}{225}$$

$$N = -\frac{1}{3K^2}\left(k_0^2 + \frac{27}{5}K^2\right)\Gamma\left(\frac{\epsilon}{2}\right) + \frac{1}{3K^2}\left(k_0^2 + 8K^2\right)\ln\left(\frac{-k^2 - i\eta}{\mu^2}\right)$$

$$+ \frac{1}{4}\left[\frac{K^4}{k_0^2} - \frac{k_0^2 k^2}{K^2}\left(14 + \frac{K^2}{k^2} + 3\frac{k^2}{K^2}\right)\right] \times D - \frac{13}{15}\ln\frac{K^2}{\mu^2}$$

$$+ \frac{1}{4}\left[\frac{K^3}{k_0^3} - \frac{k_0 k^2}{K^3}\left(14 + \frac{K^2}{k^2} + 3\frac{k^2}{K^2}\right)\right]\ln\frac{k_0 + K - i\eta}{k_0 - K + i\eta} \times \ln\frac{K^2}{(-k^2 - i\eta)}$$

$$+ \frac{1}{2K^2}\left(9k^2 + 6k_0^2 + K^2 + 3\frac{k^4}{K^2} - \frac{K^4}{k_0^4}\right)\ln\frac{K^2}{(-k^2 - i\eta)}$$

$$+ \frac{\ln 2}{K^2}\left(9k^2 + 6k_0^2 + 3\frac{k^4}{K^2} - \frac{K^4}{k_0^2} - \frac{11}{15}K^2\right)$$

$$- 16\frac{k_0^2}{K^2} + 10 + \frac{2}{9}$$

HOW TO CHECK THESE RESULTS?

't HOOFT IDENTITY

$$k_0^2\,\Gamma^{A_0 A_0} - 2k_0 K_i\,\Gamma^{A_i A_0} + K_i K_j\,\Gamma^{A_i A_j} = 0$$

$$k_\mu k_\nu\,\Gamma^{\mu\nu} \equiv 0$$

$A_i A_0$ TRANSITION

$$\Gamma^{A_i A_0} = c\, k_0 K_i \times Z$$

$$Z = -\frac{1}{3}\Gamma\left(\frac{\epsilon}{2}\right)\left(\frac{-k^2 - i\eta}{\mu^2}\right)^{-\frac{\epsilon}{2}}$$

$$+ \frac{k^2}{2K^2}\left(2k_0^2 + k^2\right) \times D$$

$$+ \frac{k^2}{2k_0 K^3}\left(2k_0^2 + k^2\right) \ln \frac{k_0 + K - i\eta}{k_0 - K + i\eta} \times \ln \frac{K^2}{(-k^2 - i\eta)}$$

$$-3\frac{k^2}{K^2} \ln \frac{K^2}{(-k^2 - i\eta)}$$

$$-6\frac{k^2}{K^2}\ln 2 + 6\frac{k_0^2}{K^2} - \frac{53}{9}$$

FORM 7×7 MATRIX OF ALL TWO-POINT FUNCTIONS

INVERT THE MATRIX TO GET PROPAGATORS

D_{00} PROPAGATOR

$$D^{A_0A_0} = \frac{1}{K^4}\left[\Gamma^{A_0A_0} + iK_n\Gamma^{A_0E_n}\right]$$

$$+ \frac{iK_m}{K^4}\left[\Gamma^{E_mA_0} + iK_n\Gamma^{E_mE_n}\right]$$

$$D^{A_0A_0} = c\,(k^2)^{-2} \times \Big\{ \frac{11}{3}\Gamma\left(\frac{\epsilon}{2}\right)K^2 - \frac{5}{3}K^2 \ln\frac{(-k^2-i\eta)}{\mu^2}$$

$$+ \frac{1}{2}k^2(k^2+2k_0^2)\times D \quad - 2K^2 \ln\frac{K^2}{\mu^2}$$

$$+ \frac{k^2}{2k_0K}(k^2+2k_0^2)\ln\frac{k_0+K-i\eta}{k_0-K+i\eta} \times \ln\frac{K^2}{(-k^2-i\eta)}$$

$$-(3k_0^2-K^2)\ln\frac{K^2}{(-k^2-i\eta)} \quad -(6k_0^2+2K^2)\ln 2$$

$$+ 6k_0^2 + \frac{31}{9}K^2 \Big\}$$

LOOK FOR THE LIMTIS

$$k_0 \longrightarrow \infty$$

$$k_0 \longrightarrow 0$$

QUARK – ANTIQUARK POTENTIAL

D. ZWANZIGER

$$P_{Coul} \equiv \lim_{R\to\infty} \frac{V(R)}{R}$$

NON-ZERO VALUE OF P_{Coul} – SIGNAL FOR COLOR CONFINEMENT

MOMENTUM SPACE

$$V(K) = \lim_{k_0\to\infty} g^2 D^{A_0A_0}(k_0, K)$$

$$\lim_{k_0\to\infty} D^{A_0A_0}(k_0,K) = \frac{c}{K^2}\left\{\frac{11}{3}\Gamma\left(\frac{\epsilon}{2}\right) - \frac{11}{3}\ln\frac{K^2}{\mu^2} - i\pi - \frac{28}{3}\ln 2 + \frac{103}{9} - 2\ln\frac{K}{k_0}\right\}$$

LIMIT $k_0 \to \infty$ DOES NOT EXIST !

J.C.TAYLOR $\qquad k_0 \to 0$

$$\lim_{k_0\to 0} D^{A_0A_0}(k_0,K) = \frac{c}{K^2}\left\{\frac{11}{3}\Gamma\left(\frac{\epsilon}{2}\right) - \frac{11}{3}\ln\frac{K^2}{\mu^2} + \frac{31}{9}\right\}$$

EVALUATE THE POTENTIAL

APPENDIX B: ONE-LOOP EXPANSION

In this appendix we find the one-loop expansion of the quantities V_0 and P_0 defined in Eqs. (28) and (29), and which appear in $D_{0,44}=V_0+P_0$. The Faddeev-Popov operator is written $M(A^{\rm tr})=M_0+M_1(A^{\rm tr})$, where $M_0\equiv-\partial_i^2$ is the negative of the Laplacian, and $(M_1)^{ac}\equiv-g_0 f^{abc}A_i^{b,\rm tr}\partial_i$. The color-Coulomb potential energy functional $\mathcal{V}(A^{\rm tr})$, defined in Eq. (25), reads

$$\mathcal{V}(\vec{x},\vec{y};A^{\rm tr})=[(M_0+M_1)^{-1}M_0(M_0+M_1)^{-1}]_{\vec{x},\vec{y}},\quad \text{(B1)}$$

and has the expansion

$$\mathcal{V}(\vec{x},\vec{y};A^{\rm tr})=[M_0^{-1}-2M_0^{-1}M_1M_0^{-1}+3M_0^{-1}M_1M_0^{-1}M_1M_0^{-1}+\ldots]_{\vec{x},\vec{y}},\quad \text{(B2)}$$

where $M_0^{-1}|_{\vec{x},\vec{y}}=(2\pi)^{-3}\int d^3k(\vec{k}^2)^{-1}\exp[i\vec{k}\cdot(\vec{x}-\vec{y})]=(4\pi|\vec{x}-\vec{y}|)^{-1}$. From Eq. (28) for V_0 we obtain to one-loop order

$$V_0(x-y)=g_0^2[M_0^{-1}+3M_0^{-1}\langle M_1M_0^{-1}M_1\rangle_0M_0^{-1}]_{\vec{x},\vec{y}}\times\delta(x_4-y_4),\quad \text{(B3)}$$

where we have used $\langle M_1(A^{\rm tr})\rangle=0$, which holds because $M_1(A^{\rm tr})$ is linear in $A^{\rm tr}$. The average designated by $\langle\ldots\rangle_0$ refers to the free-field average, with free-field propagators given in Eq. (30). This gives

$$V_0=V_{0,0}+V_{0,1}\quad \text{(B4)}$$

where the zero-loop piece is given explicitly by

$$V_{0,0}(x-y)\,\delta^{ae}=g_0^2M_0^{-1}(\vec{x}-\vec{y})\,\delta(x_4-y_4)\,\delta^{ae}\quad \text{(B5)}$$

and the one-loop piece by

$$V_{0,1}(x-y)\,\delta^{ae}=3g_0^4\int d^3x'd^3y'f^{abc}f^{cde}\langle A_i^{{\rm tr},b}(\vec{x}',x_4)\times A_j^{{\rm tr},d}(\vec{y}',x_4)\rangle_0M_0^{-1}(\vec{x}-\vec{x}')\times\partial_iM_0^{-1}(\vec{x}'-\vec{y}')\partial_jM_0^{-1}(\vec{y}'-\vec{y})\,\delta(x_4-y_4).\quad \text{(B6)}$$

These terms are illustrated in Fig. 1(a). In momentum space we have $V_{0,0}=g_0^2/\vec{k}^2$, and

$$V_{0,1}(|\vec{k}|)=\frac{3g_0^4N}{(\vec{k}^2)^2}(2\pi)^{-4}\int d^4p\frac{k_i(\delta_{ij}-\hat{p}_i\hat{p}_j)k_j}{(\vec{p}^2+p_4^2)(\vec{p}-\vec{k})^2}.\quad \text{(B7)}$$

The result of this integral is given in Eqs. (33) and (35).

Similarly, for P_0 given in Eq. (29), we have to one-loop order

$$P_0(x-y)\,\delta^{ad}=-g_0^2\langle(M_0^{-1}\rho_{\rm Coul}^a)(x)(M_0^{-1}\rho_{\rm Coul}^d)(y)\rangle_0,\quad \text{(B8)}$$

where $\rho_{\rm Coul}^a=-g_0f^{abc}A_i^{{\rm tr},b}E_i^{{\rm tr},c}$. This gives

$$P_0(x-y)\,\delta^{ad}=-g_0^2\int d^3x'd^3y'M_0^{-1}(\vec{x}-\vec{x}')\times\langle\rho_{\rm Coul}^a(\vec{x}',x_4)\rho_{\rm Coul}^d(\vec{y}',y_4)\rangle_0M_0^{-1}(\vec{y}'-\vec{y}),\quad \text{(B9)}$$

where

$$\langle\rho_{\rm Coul}^a(x)\rho_{\rm Coul}^d(y)\rangle_0=g_0^2f^{abc}f^{deg}[\langle A_i^{{\rm tr},b}(x)A_j^{{\rm tr},e}(y)\rangle_0\langle E_i^{{\rm tr},c}(x)E_j^{{\rm tr},g}(y)\rangle_0+\langle A_i^{{\rm tr},b}(x)E_j^{{\rm tr},g}(y)\rangle_0\langle E_i^{{\rm tr},c}(x)A_j^{{\rm tr},e}(y)\rangle_0].\quad \text{(B10)}$$

This term is illustrated in Fig. 1(b). In momentum space it is given by

$$P_{0,1}(k)=\frac{-g_0^4N}{(\vec{k}^2)^2}(2\pi)^{-4}\int d^4p\frac{P_{ij}(\vec{p})}{(\vec{p}^2+p_4^2)}\times\frac{P_{ij}(\vec{p}-\vec{k})}{[(\vec{p}-\vec{k})^2+(p_4-k_4)^2]}[\vec{p}^2-p_4(p_4-k_4)],\quad \text{(B11)}$$

where $P_{ij}(\vec{p})=\delta_{ij}-\hat{p}_i\hat{p}_j$ is the 3-dimensionally transverse projector. The contraction in the numerator gives 2 terms,

$$P_{ij}(\vec{p})P_{ij}(\vec{p}-\vec{k})=J_1+J_2\quad \text{(B12)}$$

$$J_1=2\quad \text{(B13)}$$

$$J_2=-\frac{\vec{p}^2\vec{k}^2-(\vec{p}\cdot\vec{k})^2}{\vec{p}^2(\vec{p}-\vec{k})^2}.\quad \text{(B14)}$$

Each term results in a Feynman integral I_1 and I_2. The integral I_2 looks more complicated. However, it is only logarithmically divergent by power counting, and when the integration is performed, the coefficient of the divergent part of I_2 vanishes, so I_2 is finite. As a result $I_2(|\vec{k}|,k_4)$ vanishes in the limit $k_4\to\infty$, and does not contribute to $P_0^{\rm as}(\vec{k})$. The result of the I_1 integration is given in Eqs. (34) and (36).

The integrals (B7) and (B11) are evaluated by dimensional regularization, with $p_4\to p_d$, and $\vec{p}=(p_i)$ for $i=1,\ldots,(d-1)$.

or explicitly

$$
\begin{aligned}
&D^{A_iA_j}\\
&=\frac{c}{k^2+\mathrm{i}\eta}\left(\delta_{ij}-\frac{K_iK_j}{K^2}\right)\\
&\quad\times\left\{\Gamma\left(\frac{\epsilon}{2}\right)-\frac{4}{3}\ln\frac{K^2}{\mu^2}+\frac{1}{3}\ln\left(\frac{-k^2-\mathrm{i}\eta}{\mu^2}\right)\right.\\
&-\frac{K^2}{4}\left[\frac{K^2+k_0^2}{k_0^2}+\frac{k_0^2}{K^2}\left(14+3\frac{k^2}{K^2}\right)\right]\times D\\
&-\frac{K}{4k_0}\left[\frac{K^2+k_0^2}{k_0^2}+\frac{k_0^2}{K^2}\left(14+3\frac{k^2}{K^2}\right)\right]\ln\frac{k_0+K-\mathrm{i}\eta}{k_0-K+\mathrm{i}\eta}\\
&\quad\times\ln\frac{K^2}{(-k^2-\mathrm{i}\eta)}\\
&+\frac{1}{2}\left(15+3\frac{k^2}{K^2}+\frac{K^2}{k_0^2}\right)\ln\frac{K^2}{(-k^2-\mathrm{i}\eta)}\\
&\left.+\left(3\frac{k^2}{K^2}+\frac{K^2}{k_0^2}+\frac{37}{3}\right)\ln 2-\frac{92}{9}\right\}. \qquad (33)
\end{aligned}
$$

5 The Slavnov–Taylor identity

Although ghosts are absent from the S-matrix elements they are necessary to formulate the Slavnov–Taylor identities [13,14]. Diagramatically they are shown for the self-energy in Fig. 8. Algebraically they are

$$k_0\Gamma^{A_0A_j}-K_i\Gamma^{A_iA_j}=(K^2\delta_{ij}-K_iK_j)\Gamma^{CA_i}. \qquad (34)$$

The diagrams involving ghost–source vertices on the right-hand side are shown in Fig. 9a,b. The diagram in Fig. 9a vanishes as the energy divergence in p_0. The diagram in Fig. 9b contributes

$$
\begin{aligned}
\Gamma^{CA_i}&=-2c\left(\frac{K^2}{\mu^2}\right)^{-\frac{\epsilon}{2}}\\
&\quad\times K_i\left\{-\frac{4}{3}+2^{-\epsilon}\Gamma\left(\frac{\epsilon}{2}\right)\left(\frac{2}{3}+\frac{13\epsilon}{9}\right)\right\}, \qquad (35)
\end{aligned}
$$

so the identity is satisfied trivially as implied by (26) and (27).

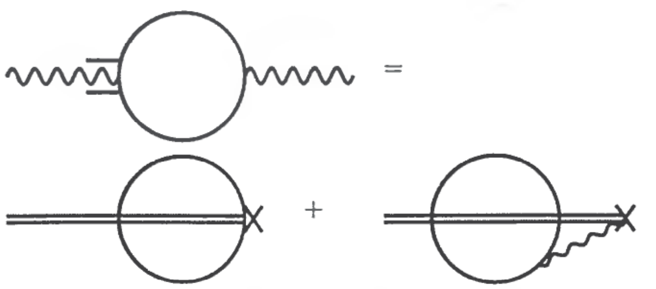


Fig. 8. The Slavnov–Taylor identity for self-energy graphs. The wavy lines stand for Yang–Mills particles and double lines for ghosts. The symbol on the left wavy line stands for the replacement of a polarization vector $e_\mu(k)$ by k_μ and k^2 need not be zero. The cross denotes the action of the tensor $(k_\mu k_\nu-k^2\delta_{\mu\nu})$. The circle represents the set of all relevant Feynman graphs

6 Discussion

We have checked the consistency of the Coulomb gauge to order g^2 including finite parts. The time-time component of the gluon propagator in the Coulomb gauge is believed to provide a long-range confining force. There are two interesting limits of (31). In the Zwanziger picture [8] g^2D_{00} gives the instantaneous part $V_Z(R)$, which is called the color-Coulomb potential. (Here D_{00} is the time-time component of the gluon propagator.) The instantaneous color-Coulomb potential $V_Z(R)$ at large R may serve as an order parameter. We have

$$K_{\mathrm{Coul}}\equiv\lim_{R\to\infty}\frac{V_Z(R)}{R}. \qquad (36)$$

A non-zero value of K_{Coul} would be the signal for color confinement. The potential is separated out in momentum space by

$$V_Z(K)=\lim_{k_0\to\infty}g^2D^{A_0A_0}(k_0,K), \qquad (37)$$

where we have written $V_Z(K)$ for the Fourier transform of $V_Z(R)$. The limit $k_0\to\infty$ of (31) is

$$
\begin{aligned}
&\lim_{k_0\to\infty}D^{A_0A_0}(k_0,K)\\
&=\frac{c}{K^2}\left\{\frac{11}{3}\Gamma\left(\frac{\epsilon}{2}\right)-\frac{11}{3}\ln\frac{K^2}{\mu^2}-\mathrm{i}\pi\right.\\
&\qquad\left.-\frac{28}{3}\ln 2+\frac{103}{9}-2\ln\frac{K}{k_0}\right\}, \qquad (38)
\end{aligned}
$$

and it is not independent of k_0. There appears to be a difference in the dominant term in (38) and (37) of Cucchieri and Zwanziger [15]. This difference arises because of the statement near the end of Appendix B in [15] that I_2 is finite, and "as a result" I_2 vanishes in the limit $k_0\to\infty$. However, the finiteness of I_2 does not imply anything about the behavior as $k_0\to\infty$. In fact, on calculating I_2, we find that the dominant term as $k_0\to\infty$ is $-4/3\ln(k_0^2/K^2)$. With this value, there is no contradiction with (38) in this paper.

Although the limit as $k_0\to\infty$ is not finite, Cucchieri and Zwanziger [15] have argued that an unambiguous instantaneous part may be defined by using renormalization group arguments.

The limit $k_0\to 0$ is naturally related to the definition of the quark–antiquark potential. It follows from considering a rectangular Wilson loop with sides of length T in

a b

Fig. 9. a Diagram with an open ghost line. The source v_n of the E_m field has the vertex $gf^{abc}E_n^bC^cv_n$. The ghost propagator is $\frac{1}{K^2}$ and it is represented with the double line. **b** Diagram with an open ghost line. The source u_i^a of the transverse gluon field has the vertex $gf^{abc}\delta_{ij}$

$$D_{00} = \frac{i}{K^2}\left[1 + c\left(\frac{11}{3}\Gamma\left(\frac{\epsilon}{2}\right) - \frac{11}{3}\ln\frac{K^2}{\mu^2} + \frac{31}{9}\right)\right]$$

$$c = \frac{g_B^2}{16\pi^2} C_G \delta_{aG}$$

$$g_B = \left(1 - \frac{11}{3}\frac{c}{\epsilon}\right) g_R \mu^{\frac{\epsilon}{2}}$$

$$V(L) = g_B^2 \int d^{3-\epsilon}K \; e^{iKL\cos\theta} \frac{i}{K^2} \times \frac{(i)^2}{i}$$

$$\times\left[1 + c\left(\frac{11}{3}\Gamma\left(\frac{\epsilon}{2}\right) - \frac{11}{3}\ln\frac{K^2}{\mu^2} + \frac{31}{9}\right)\right]$$

INSERT g_B

$$g_B^2 = \left(1 - 2\times\frac{11}{3}\frac{c}{\epsilon}\right) g_R^2 \mu^{\epsilon}$$

$$V(L) = -\left(1 - 2\times\frac{11}{3}\frac{c}{\epsilon}\right) g_R^2 \mu^{\epsilon} \times 2\pi \int_0^\infty dK\, K^{-\epsilon} \times 2\frac{\sin KL}{KL}$$

$$\times\left[1 + c\left(\frac{11}{3}\Gamma\left(\frac{\epsilon}{2}\right) - \frac{11}{3}\ln\frac{K^2}{\mu^2} + \frac{31}{9}\right]\right.$$

$$\left(\frac{\mu}{K}\right)^{\epsilon} = 1 + \epsilon\ln\frac{\mu}{K}$$

$$V(L) = -4\pi\left\{1 + \epsilon\ln\frac{\mu}{K} - 2\times\frac{11}{3}\frac{c}{\epsilon}\left(1 + \epsilon\ln\frac{\mu}{K}\right)\right\} g_R^2$$

$$\times\int_0^\infty dK \frac{\sin KL}{KL}\left[1 + c\left(\frac{11}{3}\Gamma\left(\frac{\epsilon}{2}\right) - \frac{11}{3}\ln\frac{K^2}{\mu^2} + \frac{31}{9}\right)\right]$$

KEEP TERMS TO ORDE c, i.e. g_R^2

31

$$V(L) = -4\pi g_R^2 \int_0^\infty dK \, \frac{\sin KL}{KL} \Big\{ 1 + \epsilon \ln \frac{\mu}{K} - 2 \times \frac{11}{3} \frac{c}{\epsilon} - \frac{11}{3} \ln \frac{\mu^2}{K^2}$$

$$+ c \left(1 + \epsilon \ln \frac{\mu}{K}\right) \left(\frac{11}{3} \Gamma\left(\frac{\epsilon}{2}\right) - \frac{11}{3} \ln \frac{K^2}{\mu^2} + \frac{31}{9} \right) \Big\}$$

$$V(L) = -4\pi g_R^2 \int_0^\infty dK \, \frac{\sin KL}{KL} \left\{ 1 - \frac{11}{3} c \gamma - \frac{11}{3} c \ln \frac{K^2}{\mu^2} + \frac{31}{9} c \right\}$$

$$\int_0^\infty \frac{\sin ax}{x} dx = \frac{\pi}{2} \quad \text{for } a > 0$$

$$\int_0^\infty \ln x \sin ax \, \frac{dx}{x} = -\frac{\pi}{2} (\gamma + \ln a) \quad \text{for } a > 0$$

$$V(L) = -2\pi^2 g_R^2(\mu) \frac{1}{L} \left\{ 1 + \frac{31}{9} c + \frac{11}{3} c \gamma + \frac{11}{3} c \ln (\mu L)^2 \right\}$$

ASSUME $\quad L \times \mu = 1$

$$g_R(\mu) = g_R\left(\frac{1}{L}\right)$$

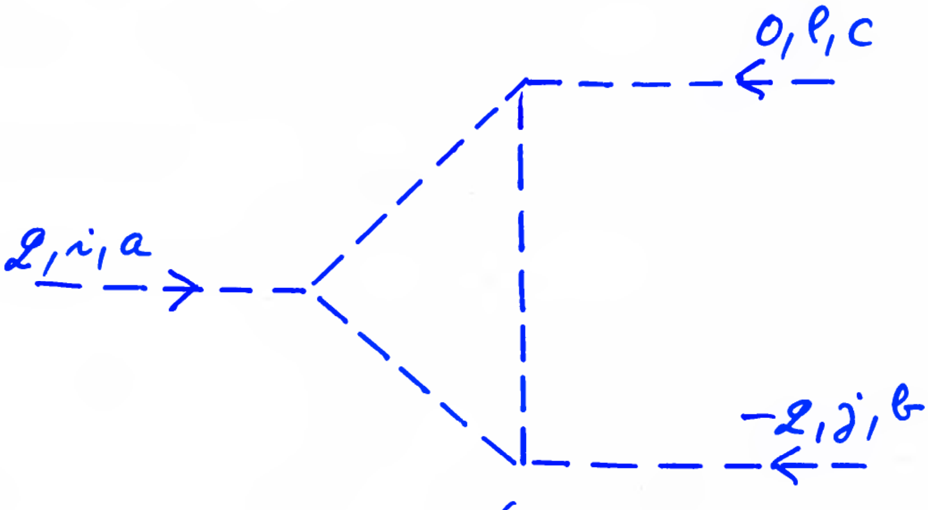
SUPPOSE $\quad g_R\left(\frac{1}{L}\right) \longrightarrow 0 \quad \text{for } L \to 0$

$$g_R\left(\frac{1}{L}\right) \longrightarrow \infty \quad \text{for } L \to \infty$$

AND MAKE EVERYBODY HAPPY!

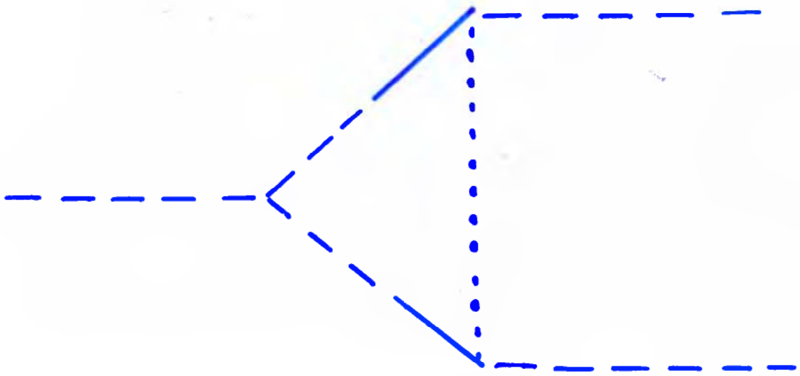
THREE-POINT FUNCTIONS

$A_i A_j A_e$ vertex



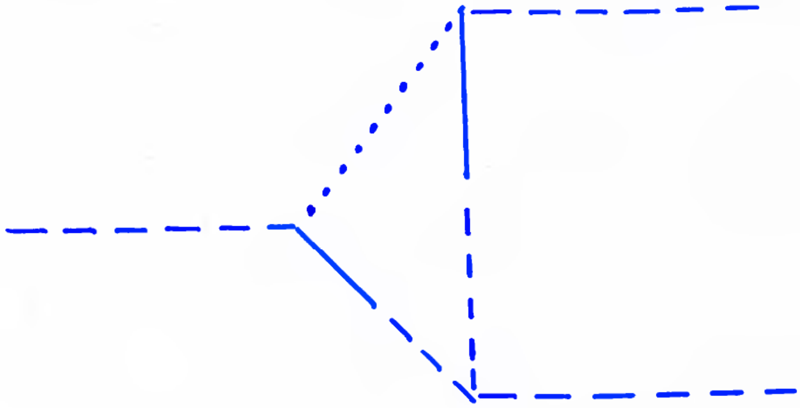
(m1)

$$(m1)^{abc}_{ije}(q,-q,0) = \frac{1}{3}\left(2q_e\delta_{ij} - q_j\delta_{ei} - q_i\delta_{ej}\right)\Gamma\left(\frac{\epsilon}{2}\right) \times g^3\pi^2 C_G f^{abc}$$



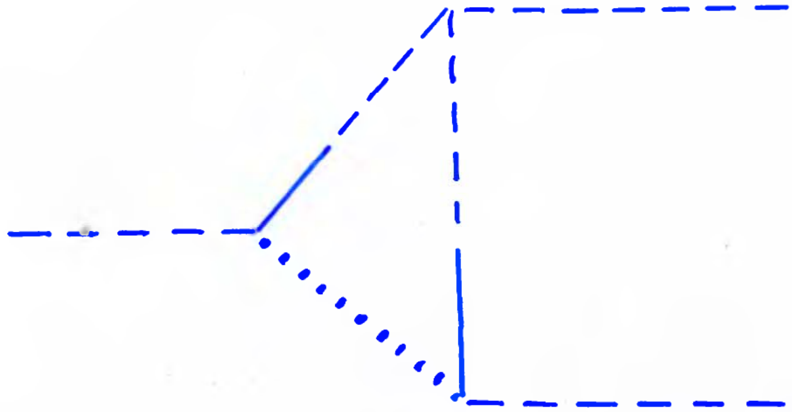
(m2a)

$$(m2a)^{abc}_{ije}(q,-q,0) = \frac{1}{30}\left(17q_e\delta_{ij} - 13q_j\delta_{ei} - 8q_i\delta_{je}\right) \times \Gamma\left(\frac{\epsilon}{2}\right) g^3\pi^2 C_G f^{abc}$$



(m2b)

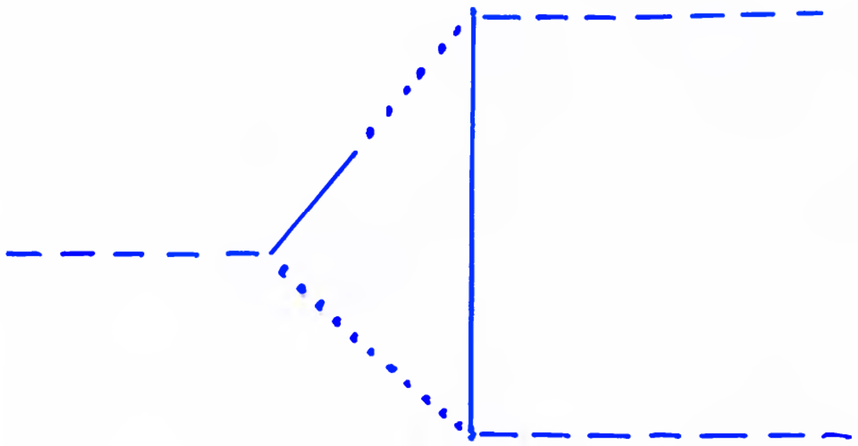
$$(m2b)^{abc}_{ije}(q,-q,0) = \frac{1}{30}(17 q_e \delta_{ij} - 13 q_i \delta_{ej} - 8 q_j \delta_{ei}) \times \Gamma\left(\frac{\epsilon}{2}\right) g^3 \pi^2 C_G f^{abc}$$



(m2c)

$$(m2c)^{abc}_{ije}(q,-q,0) = \frac{2}{3}\left\{q_e \delta_{ij} - \frac{1}{5}(q_e \delta_{ij} + q_j \delta_{ei} + q_i \delta_{ej})\right\} \times \Gamma\left(\frac{\epsilon}{2}\right) g^3 \pi^2 C_G f^{abc}$$

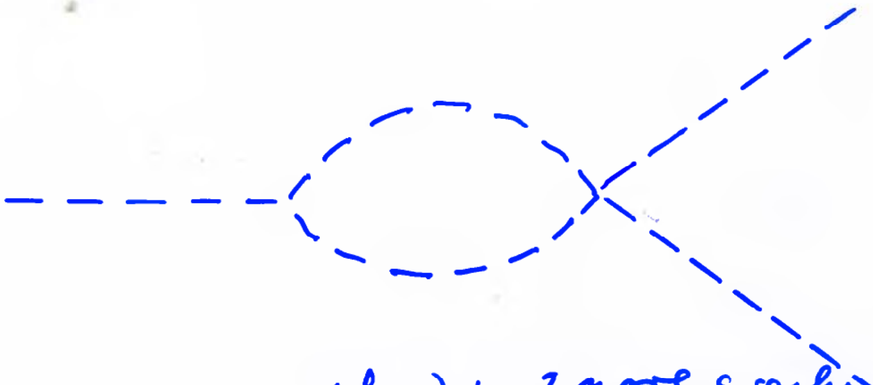
$$(m2)^{abc}_{ije\,class} = -\frac{5}{6}(q_i \delta_{je} + q_j \delta_{ei} - 2 q_e \delta_{ij}) \times \Gamma\left(\frac{\epsilon}{2}\right) g^3 \pi^2 C_G f^{abc}$$



$(m1a) + (5 \text{ more graphs})$

$$(m1)^{abc}_{ijl}(q,-q,0) = \frac{2}{3}\left(2q_l\delta_{ij} - q_i\delta_{jl} - q_j\delta_{li}\right) \times \Gamma\left(\frac{\epsilon}{2}\right) g^3\pi^2 C_G f^{abc}$$

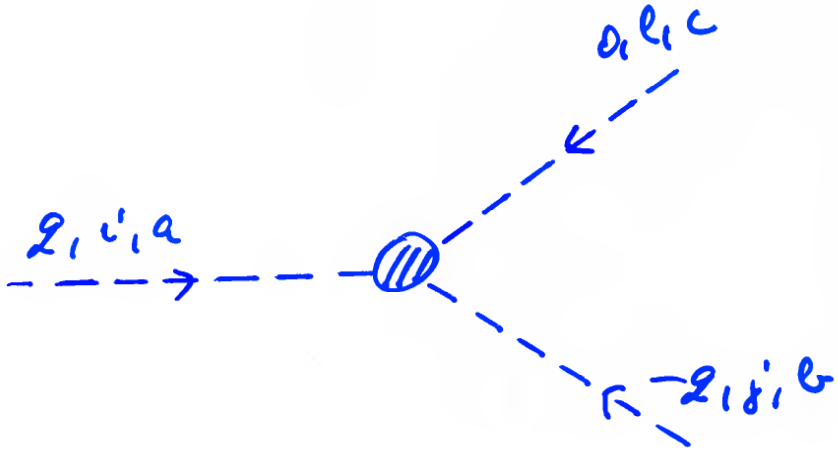
4-GLUON COUPLING



$(fa) + 2 \text{ more graphs}$

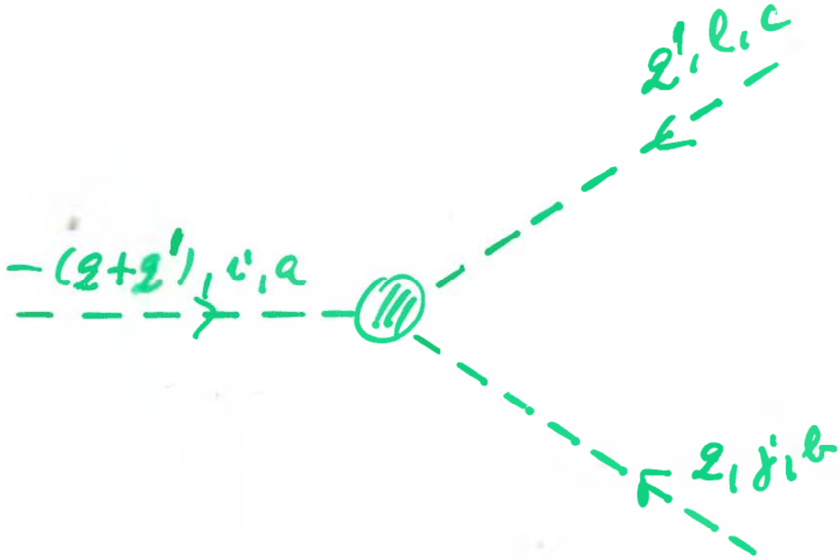
$$(f)^{abc}_{ijl}(q,-q,0) = \frac{3}{2}\left(q_i\delta_{jl} + q_j\delta_{il} - 2q_l\delta_{ij}\right) \times \Gamma\left(\frac{\epsilon}{2}\right) g^3 C_G \pi^2 f^{abc}$$

$A_i A_j A_l$ COUNTER-TERM



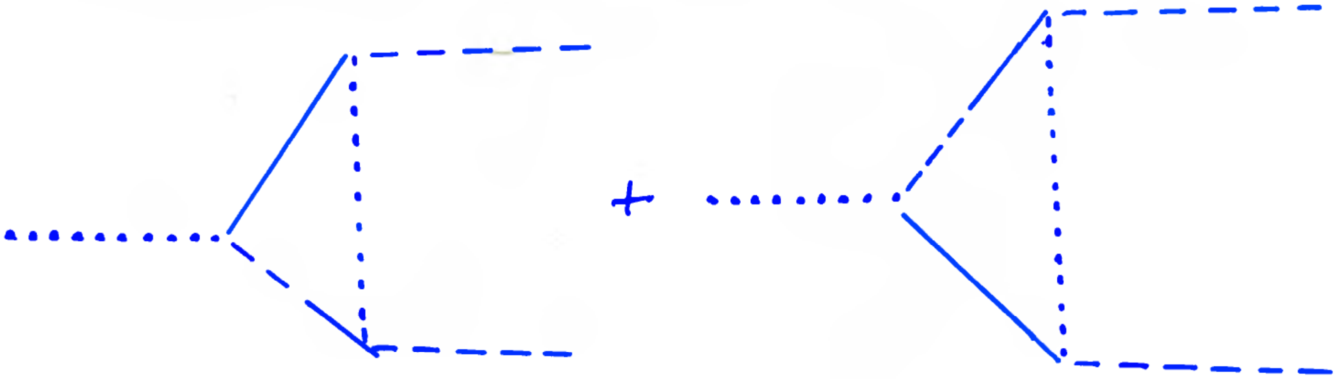
$$= \frac{1}{3}\left(2q_l\delta_{ij} - q_i\delta_{jl} - q_j\delta_{li}\right)\Gamma\left(\frac{\epsilon}{2}\right) g^3 \pi^2 C_A f^{abc}$$

GENERAL MOMENTA

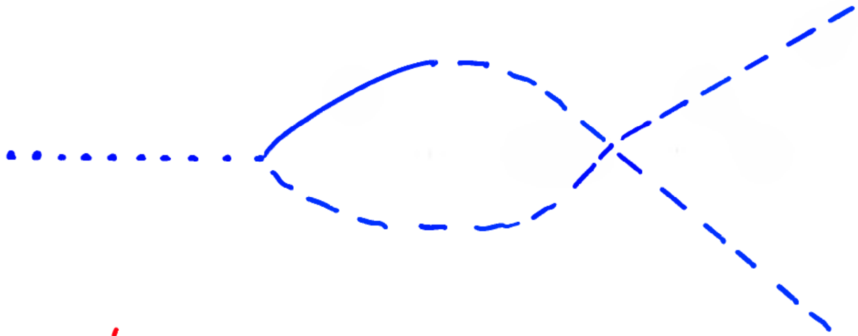


$$= -\frac{1}{3}\left[(2q+q')_l\delta_{ij} + (q'-q)_i\delta_{jl} - (2q'+q)_j\delta_{li}\right] \cdot \Gamma\left(\frac{\epsilon}{2}\right) g^3 \pi^2 C_A f^{abc}$$

$A_0 A_i A_j$ VERTEX NOT IN ORIGINAL LAGRANGIAN



$= 0$



$= 0$

BUT !

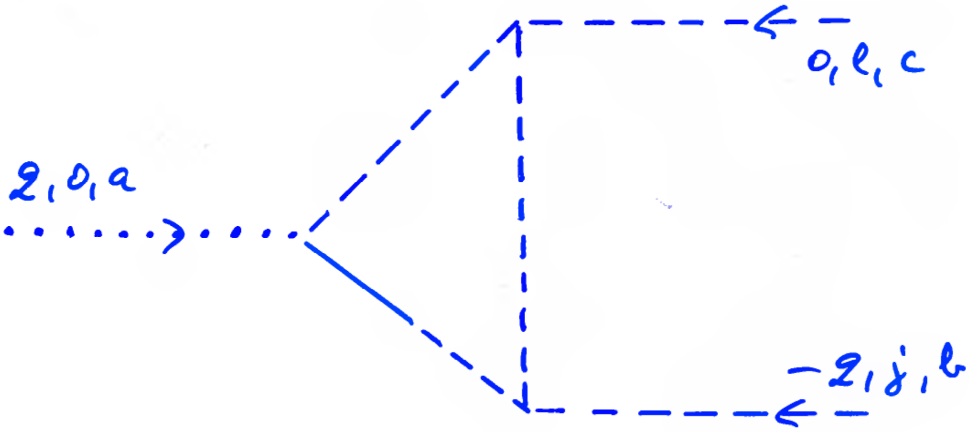


Fig. (x)

$$(x)^{abc}_{0je}(q,-q,0) = -\frac{2}{3}\, q_0\, \delta_{je}\, \Gamma\!\left(\frac{\epsilon}{2}\right) g^3 \pi^2 C_a f^{abc}$$

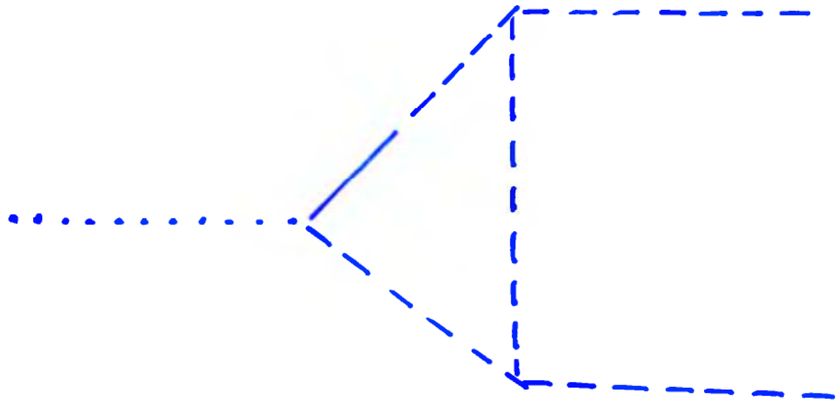
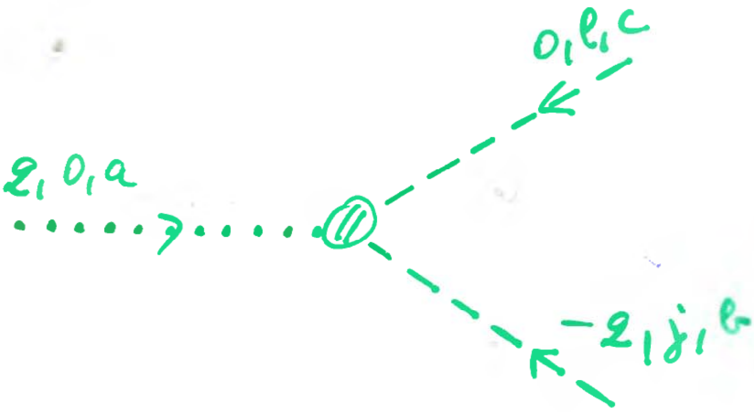


Fig. (y)

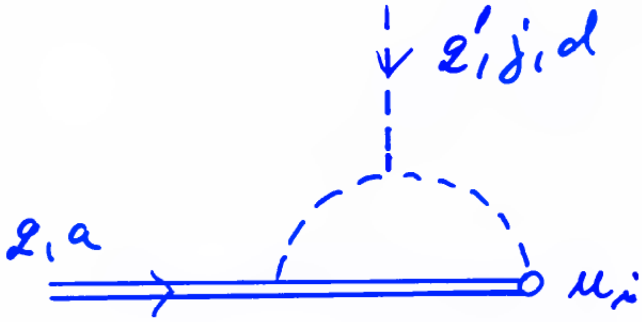
$$(y)^{abc}_{0j\ell}(q,-q,0) = \frac{1}{3} q_0 \delta_{j\ell}\, \Gamma\left(\frac{\epsilon}{2}\right) g^3 \pi^2 C_A f^{abc}$$

$A_0 A_j A_\ell$ COUNTER-TERM

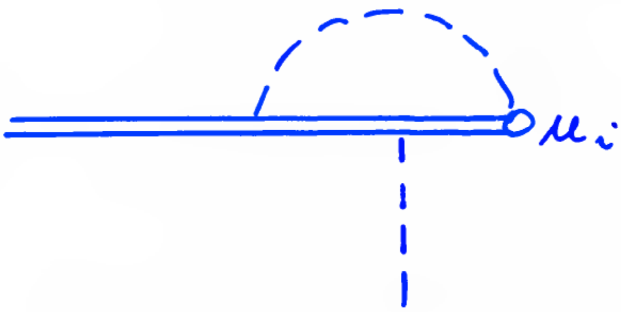


$$= -\frac{1}{3} q_0 \delta_{j\ell}\, \Gamma\left(\frac{\epsilon}{2}\right) g^3 \pi^2 C_A f^{abc}$$

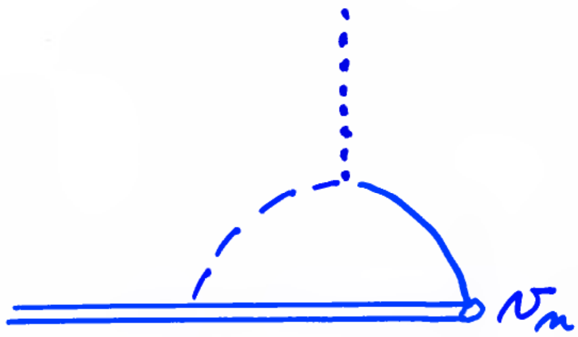
GHOST GRAPHS



$= 0$

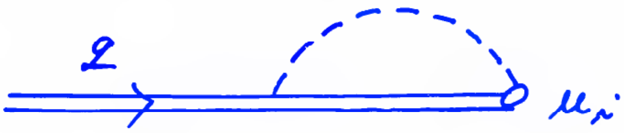


$= 0$



$= 0$

$= 0$



$$(cA_i)^{ab} = -\frac{4}{3} q_i \,\Gamma\left(\frac{\epsilon}{2}\right) \frac{ig^2}{16\pi^2} C_G \,\delta_{ab}$$

GENERATING FUNCTIONAL Γ SATISFIES BRS

$$\int dx \left[\frac{\delta\Gamma}{\delta A_i}\cdot\frac{\delta\Gamma}{\delta u_i} + \frac{\delta\Gamma}{\delta A_0}\frac{\delta\Gamma}{\delta u_0} + \frac{\delta\Gamma}{\delta E}\cdot\frac{\delta\Gamma}{\delta v} \right] = 0$$

TO 1-LOOP ORDER

$$\Gamma = \Gamma_0 + \Gamma_1$$

Γ_0 – SATISFIES BRS BY ITSELF

FROM THE GHOST GRAPH Γ_1 CONTAINS A TERM OF THE FORM

$$s\, u\, \partial c$$

$$s = -\frac{4}{3}\Gamma\left(\frac{\epsilon}{2}\right)\frac{ig^2}{16\pi^2} C_G$$

TO 1-LOOP ORDER BRS CONTAINS ONE Γ_0 & ONE Γ_1.

$$\int dx \left[\frac{\delta\Gamma_1}{\delta A_i}\delta A_i + \frac{\delta\Gamma_1}{\delta A_0}\delta A_0 + \frac{\delta\Gamma_1}{\delta E}\delta E + \frac{\delta\Gamma_0}{\delta A_i}\cdot(s\,\partial_i c) \right] = 0$$

GHOST CONTRIBUTION IN MOMENTUM SPACE (

$$\frac{\delta \Gamma_0}{\delta A_i}(s\partial_i C) = -\frac{4}{3}\Gamma\left(\frac{\epsilon}{2}\right)\frac{ig^2}{16\pi^2} C_G\, q_i$$

$$\times(-ig) f^{abc}\left[-2q_e\delta_{ij} + q_i\delta_{ej} + q_j\delta_{ei}\right]$$

$$= igf^{abc}\,\frac{4}{3}\left(Q^2\delta_{ej} - q_e q_j\right)\Gamma\left(\frac{\epsilon}{2}\right)\frac{i\pi^2}{(2\pi)^4}\, g^2 C_G$$

GLUON SELF-ENERGY

$$\Pi^{ab}_{je}(q) = \left[q_j q_e - Q^2\delta_{je} - \frac{1}{3}q_0^2\delta_{je}\right]\Gamma\left(\frac{\epsilon}{2}\right) g^2 C_G\,\delta_{ab}\,\frac{i\pi^2}{(2\pi)^4}$$

SLAVNOV–TAYLOR ID. FOR EFFECTIVE ACTION

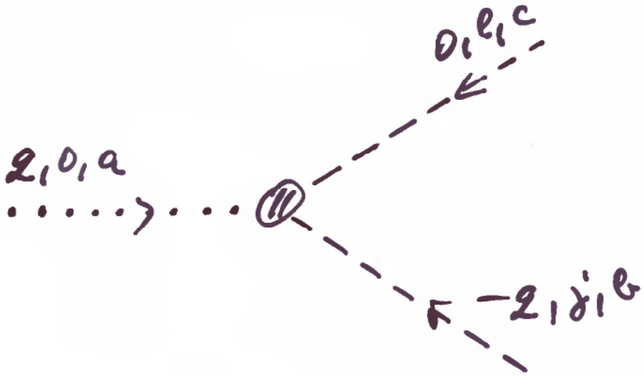
$$q_0 V_{0je}(q,-q,0) - q_i V_{ije}(q,-q,0) + \text{ghost}$$

$$= -g\,\Pi_{je}(-q)$$

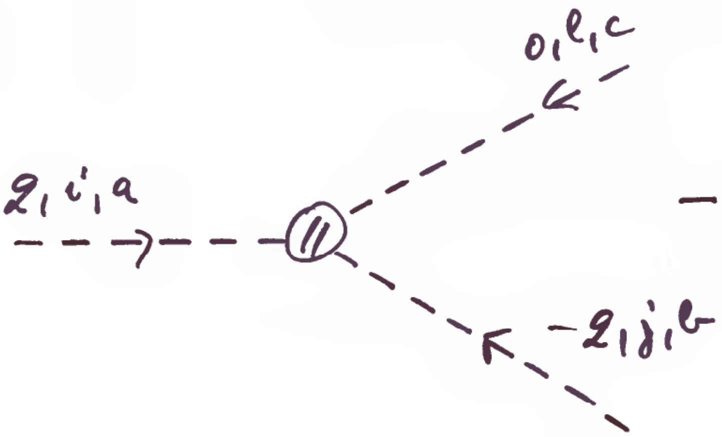
WHERE THE 3-GLUON TERM IN THE EFFECTIVE ACTION WILL BE DENOTED BY

$$if^{abc}\,V_{\mu\nu\lambda}(p,q,p')$$

MOMENT OF TRUTH



$$\frac{1}{3} q_0 \delta_{je} \Gamma\left(\frac{\epsilon}{2}\right) i f^{abc} \frac{i\pi^2}{(2\pi)^4} g^3 C_G$$



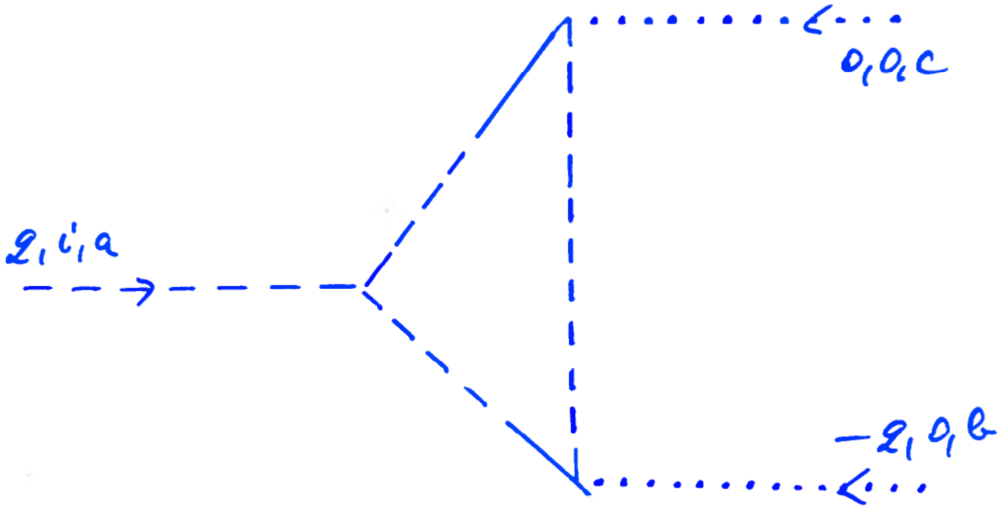
$$-\frac{1}{3}\left(2 q_e \delta_{ij} - q_i \delta_{ej} - q_j \delta_{ei}\right)$$
$$\times\, i f^{abc} \frac{i\pi^2}{(2\pi)^4} g^3 C_G \Gamma\left(\frac{\epsilon}{2}\right)$$

OMIT WRITING COMMON FACTOR

$$\frac{1}{3} q_0^2 \delta_{je} - q_i \left(-\frac{1}{3}\right)\left(2 q_e \delta_{ij} - q_i \delta_{ej} - q_j \delta_{ei}\right)$$
$$+ \frac{4}{3}\left(Q^2 \delta_{ej} - q_e q_j\right)$$
$$= \frac{1}{3} q_0^2 \delta_{ej} - \left(q_e q_j - Q^2 \delta_{ej}\right)$$

S−T id. satisfied!

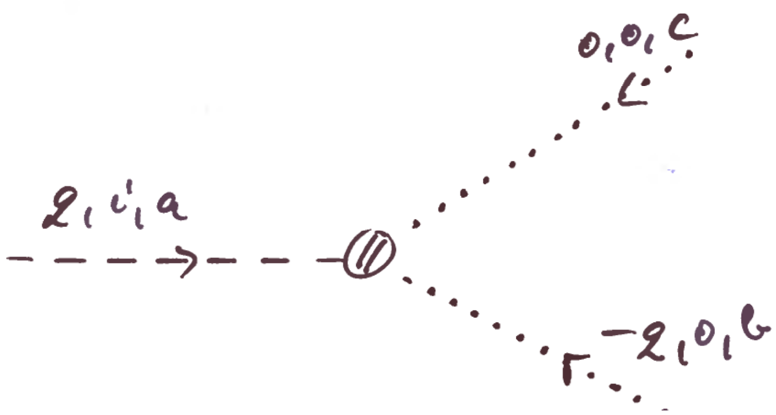
$(A_i A_0 A_0)$ VERTEX — NOT IN THE ORIGINAL LAGRANGIAN



(m5)

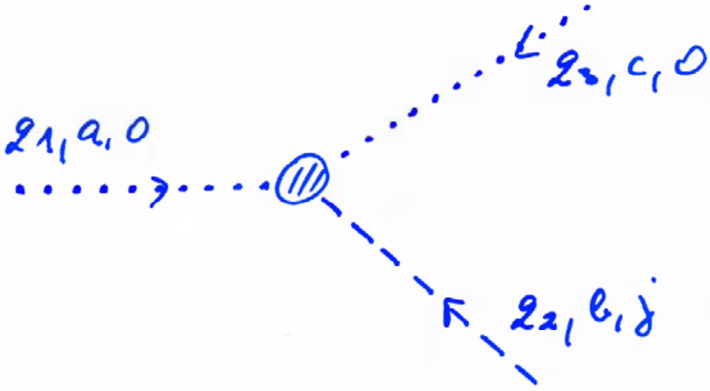
$$(m5)^{abc}_{i00}(q,-q,0) = \frac{1}{12}\, q_i\, \Gamma\!\left(\frac{\epsilon}{2}\right) g^3 \pi^2 C_G f^{abc}$$

8 such graphs

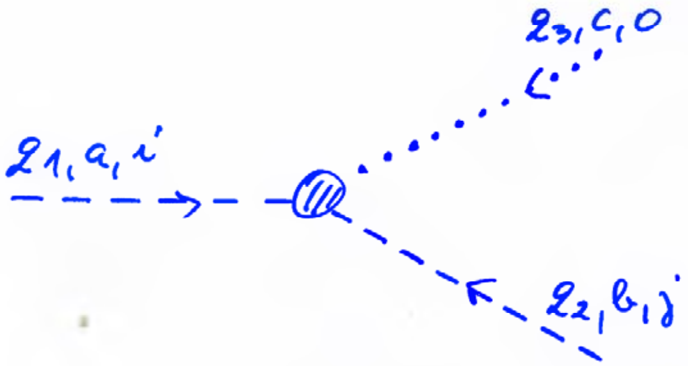


$$\frac{1}{3}\, q_i\, \Gamma\!\left(\frac{\epsilon}{2}\right) g^3 \pi^2 C_G f^{abc}$$

GENERAL MOMENTA



$$V^{abc}_{0j0}(q_1,q_2,q_3) = \frac{1}{3}(q_1-q_3)_j\, \Gamma\left(\frac{\epsilon}{2}\right) g^3\pi^2 C_A f^{abc}$$



$$V^{abc}_{ij0}(q_1,q_2,q_3) = -\frac{1}{3}(q_2-q_1)_0\, \delta_{ij}\cdot \Gamma\left(\frac{\epsilon}{2}\right) g^3\pi^2 C_A f^{abc}$$

 $-\frac{1}{3} q_0 q_i\, \Gamma\left(\frac{\epsilon}{2}\right) \frac{i\pi^2}{(2\pi)^4} g^2 C_A \delta_{ab}$

S-T id.

$$(q_1)_0 V^{abc}_{0j0}(q_1,q_2,q_3) - (q_1)_i V^{abc}_{ij0}(q_1,q_2,q_3)$$
$$= -g\,[\,SE(q_2) - SE(q_3)\,]$$

CONCLUSIONS

1. KNOWLEDGE OF COVARIANT GAUGES DOES NOT HELP WITH PHYSICAL GAUGES

2. NO RELIABLE REGULARIZATION

3. EACH INTEGRAL IS A PIECE OF ART

4. D_{00} DOES NOT SHOW CONFINEMENT TO ORDER g^2

5. VERTICES WHICH DO NOT EXIST IN THE ORIGINAL LAGRANGIAN APPEAR AT HIGHER ORDERS AS COUNTER-TERMS

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