Lie algebra type noncommutative phase spaces are Hopf algebroids

Stjepan Meljanac, Zoran Škoda
meljanac@irb.hr, zskoda@irb.hr
Theoretical Physics Division, Institute Rudjer Bošković
Bijenička cesta 54, P.O.Box 180, HR-10002 Zagreb, Croatia
and Martina Stojić stojic@math.hr
Department of Mathematics, University of Zagreb,
Bijenička cesta 3, HR-10000 Zagreb, Croatia

Abstract. For a noncommutative configuration space whose coordinate algebra is the universal enveloping algebra of a finite dimensional Lie algebra, it is known how to introduce an extension playing the role of the corresponding noncommutative phase space, namely by adding the commuting deformed derivatives in a consistent and nontrivial way, therefore obtaining certain deformed Heisenberg algebra. This algebra has been studied in physical contexts, mainly in the case of the kappa-Minkowski space-time. Here we equip the entire phase space algebra with a coproduct, so that it becomes an instance of a completed variant of a Hopf algebroid over a noncommutative base, where the base is the enveloping algebra.

Keywords: universal enveloping algebra, noncommutative phase space, deformed derivative, Hopf algebroid, completed tensor product

AMS classification: 16S30,16S32, 16S35, 16Txx

1. Introduction

Recently, a number of physical models has been proposed [1, 10, 15], where the background geometry is described by a noncommutative *configuration* space of Lie algebra type. Descriptively, its coordinate algebra is the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. So-called κ-Minkowski space is the most explored example [9, 15, 16, 17]. That space has been used to build a model featuring the double special relativity, a framework modifying special relativity, proposed to explain some phenomena observed in the high energy gamma ray bursts.

The noncommutative *phase* space of the Lie algebra $\mathfrak{g}$ is introduced by enlarging $U(\mathfrak{g})$ with additional associative algebra generators, the *deformed derivatives*, which act on $U(\mathfrak{g})$ via an action $\blacktriangleright$ satisfying deformed Leibniz rules [19, 20]. The subalgebra generated by the deformed derivatives is commutative. In fact, this commutative algebra is a topological Hopf algebra isomorphic to the full algebraic dual $U(\mathfrak{g})^*$ of the enveloping algebra. In this article, we extend the coproduct of the topological Hopf algebra $U(\mathfrak{g})^*$ of deformed derivatives to a coproduct $\Delta : H \to H\hat{\otimes}_{U(\mathfrak{g})}H$ on the whole phase space H (and its completion $\hat{H}$); this coproduct is moreover a part of a Hopf algebroid structure on H over the noncommutative base algebra $U(\mathfrak{g})$. Roughly, this means that the coproduct does not take value in a tensor product $H \otimes H$ over the ground field, like for Hopf algebras, but in a tensor product of $U(\mathfrak{g})$-bimodules, where H is a bimodule with the help of so called source and target maps. The notion of a Hopf algebroid is slightly adapted regarding that the tensor product $\hat{\otimes}_{U(\mathfrak{g})}$ in the definition of the coproduct is understood in a completed sense; a part of the definition still needs the tensor products without completions. Our algebroid structure is similar but a bit weaker than the Hopf algebroid *internal* [3] to the tensor category of complete cofiltered vector spaces [18].

The noncommutative phase space of Lie type is nontrivially isomorphic to the topological Heisenberg double of $U(\mathfrak{g})$ [20]. Heisenberg doubles of finite dimensional Hopf algebras are known to carry a Hopf algebroid structure [6, 14]. However, our starting Hopf algebra $U(\mathfrak{g})$ is infinite-dimensional, though filtered by finite-dimensional pieces. While the generalities on such filtered algebras can be used to obtain the Hopf algebroid structure [18], we here use the specific features of $U(\mathfrak{g})$ instead, and in particular the matrix $\mathcal{O}$, introduced in the Section 2. From a geometric viewpoint, where $U(\mathfrak{g})$ is viewed as the algebra of left invariant differential operators on a Lie group, the matrix $\mathcal{O}$ is interpreted as a transition matrix between a basis of left invariant and a basis of right invariant vector fields. Then our phase space appears as the algebra of formal differential operators around the unit of the Lie group. A different variant of the Hopf algebroid structure has been outlined in [12, 13], for the special case when the Lie algebra is the κ-Minkowski space, at a physical level of rigor.

We assume familiarity with bimodules, coalgebras, comodules, bialgebras, Hopf algebras, Hopf pairings and the Sweedler notation for comultiplications (coproducts) $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$, and right coactions $\rho(v) = \sum v_{(0)} \otimes v_{(1)}$ (with or without the explicit summation sign). We do not assume previous familiarity with Hopf algebroids. In noncommutative geometry, one interprets Hopf algebroids [2, 4, 6, 14] as formal duals to quantum groupoids.

The generic symbols for the multiplication map, comultiplication, counit and antipode will be $m, \Delta, \epsilon, \mathcal{S}$, with various subscripts. All algebras are over a fixed ground field $\boldsymbol{k}$ of characteristic zero (in physical applications $\mathbb{R}$ or $\mathbb{C}$). The opposite algebra of an associative algebra A is denoted A^{op}, and the coopposite coalgebra to $C = (C, \Delta)$ is $C^{\mathrm{co}} = (C, \Delta^{\mathrm{op}})$. Given a vector space V, denote its algebraic dual by $V^* := \mathrm{Hom}(V, \boldsymbol{k})$, and the corresponding symmetric algebras $S(V)$ and

$S(V^*)$. If an algebra A is graded, we label its graded (homogeneous) components by upper indices, $A = \oplus_{i=0}^{\infty} A^i$, $A^i \cdot A^j \subset A^{i+j}$, and, if B is filtered, we label its filtered components $B_0 \subset B_1 \subset B_2 \subset \ldots$ by lower indices, $B_i \cdot B_j \subset B_{i+j}$ and $B = \cup_{i=0}^{\infty} B_i$. The Einstein summation convention on repeated Greek indices is assumed throughout the article.

2. Deformed phase space

Throughout the article $\mathfrak{g}$ is a fixed Lie algebra over $\boldsymbol{k}$ of some finite dimension n. In a basis $\hat{x}_1, \ldots, \hat{x}_n$ of $\mathfrak{g}$,

$$[\hat{x}_\mu, \hat{x}_\nu] = C^\lambda_{\mu\nu} \hat{x}_\lambda. \tag{1}$$

The generators of the universal enveloping algebra $U(\mathfrak{g})$ are also denoted by $\hat{x}_\mu$, unlike the generators of the symmetric algebra $S(\mathfrak{g})$ which are denoted by $x_1, \ldots, x_n$ (without hat symbol) instead. The dual basis of $\mathfrak{g}^*$ and the corresponding generators of $S(\mathfrak{g}^*)$ are denoted $\partial^1, \ldots, \partial^n$. $S(\mathfrak{g}) = \oplus_{i=0}^{\infty} S^i(\mathfrak{g}) = \cup_{i=0}^{\infty} S_i(\mathfrak{g})$ carries a graded and $U(\mathfrak{g}) = \cup_i U_i(\mathfrak{g})$ a *filtered* Hopf algebra structure. Both structures are induced along quotient maps from the tensor bialgebra $T(\mathfrak{g})$. By the PBW theorem, the linear map

$$\xi : S(\mathfrak{g}) \to U(\mathfrak{g}), \qquad x_{i_1} \cdots x_{i_r} \mapsto \frac{1}{r!} \sum_{\sigma \in \Sigma(r)} \hat{x}_{i_{\sigma(1)}} \cdots \hat{x}_{i_{\sigma(r)}}, \tag{2}$$

is an isomorphism of filtered coalgebras whose inverse may be identified with the projection to the associated graded ring [5]. When applied to spaces, we use the hat symbol $\hat{}$ for completions. Our main example is $\hat{S}(\mathfrak{g}^*) = \varprojlim_i S_i(\mathfrak{g}^*) \cong \prod_i S^i(\mathfrak{g}^*)$ which is the completion of $S(\mathfrak{g}^*)$ with respect to the degree of polynomial; it may be identified with the formal power series ring $\boldsymbol{k}[[\partial^1, \ldots, \partial^n]]$ in n variables. For our purposes, it is the same to regard this ring, as well as the algebraic duals $U(\mathfrak{g})^*$ and $S(\mathfrak{g}^*)$, either as topological or as cofiltered algebras (see Appendix A.2). For a multiindex $K = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$, denote $|K| := k_1 + \ldots + k_n$, $x_K := x_1^{k_1} \cdots x_n^{k_n}$ and $\hat{x}_K := \hat{x}_1^{k_1} \cdots \hat{x}_n^{k_n}$. The multiindices add up componentwise. The partial order on $\mathbb{N}_0^n$ induced by the componentwise $<$ is also denoted $<$. If J, K are multiindices the rule $\langle \partial^J, x_K \rangle := J!\, \delta^J_K$ continuously and linearly extends to a unique map $\langle , \rangle : \hat{S}(\mathfrak{g}^*) \otimes S(\mathfrak{g}) \to \boldsymbol{k}$ ('evaluation of a partial differential operator at a polynomial in zero'), which is a nondegenerate pairing, hence it identifies $\hat{S}(\mathfrak{g}^*) \cong S(\mathfrak{g})^*$. The map

$$\xi^T : U(\mathfrak{g})^* \to S(\mathfrak{g})^* \cong \hat{S}(\mathfrak{g}^*) \tag{3}$$

transpose (dual) to ξ (see (2)) is an isomorphism of cofiltered algebras. Introduce the opposite Lie algebra $\mathfrak{g}^R$ generated by $\hat{y}_\mu$, where

$$[\hat{y}_\mu, \hat{y}_\nu] = -C^\lambda_{\mu\nu}\hat{y}_\lambda. \tag{4}$$

The Lie algebra $\mathfrak{g}^R$ is antiisomorphic to $\mathfrak{g}^L := \mathfrak{g}$ via $\hat{y}_i \mapsto \hat{x}_i$, inducing an isomorphism $U(\mathfrak{g}^L)^{\mathrm{op}} \cong U(\mathfrak{g}^R)$. We consider spaces $\mathfrak{g}^L$ and $\mathfrak{g}^R$ distinct.

The n-th **Weyl algebra** A_n is the associative algebra generated by $x_1, \ldots, x_n$, $\partial^1, \ldots, \partial^n$ subject to relations $[x_\alpha, x_\beta] = [\partial^\alpha, \partial^\beta] = 0$ and $[\partial^\alpha, x_\beta] = \delta^\alpha_\beta$. The (semi)completed Weyl algebra $\hat{A}_n$ is the completion of A_n by the "degree of a differential operator", hence allowing the formal power series in ∂^α-s. A_n has a faithful representation, called *Fock space*, on the polynomial algebra in $x_1, \ldots, x_n$, in which each x_μ acts as a multiplication operator and ∂^μ as a partial derivative; the action of A_n extends continuously to a unique action of $\hat{A}_n$. We construct certain analogues of $\hat{A}_n$ containing $U(\mathfrak{g}^L)$ or $U(\mathfrak{g}^R)$. They have a structure of a Hopf-algebraic smash product.

DEFINITION 1. *Let A be an algebra and B a bialgebra.*

A left action $\triangleright : B \otimes A \to A$ (right action $\triangleleft : A \otimes B \to B$), is a left (right) ***Hopf action*** *if $b \triangleright (aa') = \sum (b_{(1)} \triangleright a) \cdot (b_{(2)} \triangleright a')$ and $b \triangleright 1 = \epsilon(b)1$ or, respectively, $(aa') \triangleleft b = \sum (a \triangleleft b_{(1)})(a \triangleleft b_{(2)})$ and $1 \triangleleft b = \epsilon(b)1$, for all $a, a' \in A$ and $b, b' \in B$. We then also say that A is a left (right) B-module algebra. Given a left (right) Hopf action, the* **smash product** *$A\sharp B$ ($B\sharp A$) is an associative algebra which is a tensor product vector space $A \otimes B$ ($B \otimes A$) with the multiplication bilinearly extending the formulas*

$$\begin{array}{l} (a\sharp b)(a'\sharp b') = \sum a(b_{(1)} \triangleright a')\sharp b_{(2)}b', \quad a, a' \in A, b, b' \in B, \\ (b\sharp a)(b'\sharp a') = \sum bb'_{(1)}\sharp(a \triangleleft b'_{(2)})a', \quad a, a' \in A, b, b' \in B, \end{array}$$

where, for emphasis, one writes $a\sharp b := a \otimes b$.

Note that $1\sharp A$ and $B\sharp 1$ are subalgebras in $B\sharp A$, canonically isomorphic to A and B. We may exchange the actions $\triangleright$ for the homomorphisms of algebras $\psi : B \to \mathrm{End}\, A$, $\psi(b)(a) := b \triangleright a$. If B is a Hopf algebra with an antipode $\mathcal{S}$, we may replace ψ with $\psi \circ \mathcal{S} : B^{\mathrm{co}} \to \mathrm{End}^{\mathrm{op}} A$, or a *right* Hopf action $\triangleleft : A \otimes B^{\mathrm{co}} \to A$, $\triangleleft : a \otimes b \mapsto a \triangleleft b := \mathcal{S}(b) \triangleright a$ enabling us to define $A\sharp B^{\mathrm{co}}$. If B is cocommutative (for instance, $B^{\mathrm{co}} = B = U(\mathfrak{g})$ below) then $\mathcal{S}^2 = \mathrm{id}$ and there is an isomorphism $A\sharp B \cong B\sharp A$ of algebras, $a\sharp b \mapsto \sum b_{(1)}\sharp(a \triangleleft b_{(2)})$, with inverse $b\sharp a \mapsto \sum (b_{(1)} \triangleright a)\sharp b_{(2)}$.

Given $\mathfrak{g}^L$ as above, let $\mathcal{C}$ denote the $n \times n$-matrix with entries

$$\mathcal{C}^\alpha_\beta := C^\alpha_{\beta\gamma}\partial^\gamma. \tag{5}$$

In this notation,

$$\phi := \frac{-\mathcal{C}}{e^{-\mathcal{C}}-1} = \sum_{N=0}^{\infty} \frac{(-1)^N B_N}{N!}\mathcal{C}^N \tag{6}$$

is a power series in matrix $\mathcal{C}$, which in turn contains ∂^β-s; in particular ϕ is itself a matrix whose entries $\phi^\alpha_\beta \in \hat{S}(\mathfrak{g}^*)$, $\alpha, \beta = 1, \ldots, n$, are formal power series in ∂-s. The constants B_N are the Bernoulli numbers. The formula $\boldsymbol{\phi}(\hat{x}_\alpha)(\partial^\beta) := \phi^\beta_\alpha$ defines a linear map $\boldsymbol{\phi}(\hat{x}_\alpha) : \mathfrak{g} \to \hat{S}(\mathfrak{g}^*)$, which by the chain rule and continuity extends to a unique derivation $\boldsymbol{\phi}(\hat{x}_\alpha) \in \mathrm{Der}(\hat{S}(\mathfrak{g}^*))$. A crucial property of ϕ is that the corresponding map $\boldsymbol{\phi} : \mathfrak{g}^L \to \mathrm{Der}(\hat{S}(\mathfrak{g}^*))$ is a Lie algebra homomorphism; it follows that it automatically extends to a unique right Hopf action also denoted

$$\boldsymbol{\phi} : U(\mathfrak{g}^L) \to \mathrm{End}^{\mathrm{op}}(\hat{S}(\mathfrak{g}^*)). \tag{7}$$

and the corresponding smash product $H^L := U(\mathfrak{g}^L)\sharp_{\boldsymbol{\phi}}\hat{S}(\mathfrak{g}^*)$ and interpret it as the 'noncommutative phase space of Lie type'. (Warning: in [20] we used the notation $\boldsymbol{\phi}$ for the *left* Hopf action $\boldsymbol{\phi}\circ\mathcal{S}_{U(\mathfrak{g}^L)}$, which is less fundamental.) We commonly identify $\hat{S}(\mathfrak{g}^*)$ with the subalgebra $1\sharp\hat{S}(\mathfrak{g}^*)$ and $U(\mathfrak{g}^L)$ with $U(\mathfrak{g}^L)\sharp 1$. It follows that in H^L

$$[\partial^\mu, \hat{x}_\nu] = \left(\frac{-\mathcal{C}}{e^{-\mathcal{C}}-1}\right)^\mu_\nu . \tag{8}$$

This identity justifies the interpretation of ∂^μ as deformed partial derivatives. Though we do not use this fact below, note that H^L may be constructed as the free product of $U(\mathfrak{g}^L)$ and $\hat{S}(\mathfrak{g}^*)$, quotiented by the smallest complete ideal containing the identities (8). The universal formula (6) for ϕ is, in this context, derived in [8] and H^L is studied in [19]. There is an induced monomorphism $()^\phi : U(\mathfrak{g}) \to \hat{A}_n$ of algebras which we call the ϕ-**realization** of $U(\mathfrak{g})$ (by formal differential operators) given by $\hat{x}_\nu \mapsto \hat{x}^\phi_\nu := x_\rho \phi^\rho_\nu$ on the generators and, if complemented by the formulas $\partial^\mu \mapsto \partial^\mu$, it defines a unique continuous isomorphism of algebras $U(\mathfrak{g}^L)\sharp\hat{S}(\mathfrak{g}^*) \cong \hat{A}_n$ (ϕ-realization of H^L).

Regarding that $\mathfrak{g}^R$ is the Lie algebra with known structure constants, $-C^\alpha_{\beta\gamma}$, the formula (6) can be applied to it. This yields a matrix $\tilde{\phi} := \left(\frac{\mathcal{C}}{e^{\mathcal{C}}-1}\right)$ and a left Hopf action $\tilde{\boldsymbol{\phi}} : U(\mathfrak{g}^R) \to \mathrm{End}(\hat{S}(\mathfrak{g}^*))$ hence a smash product $H^R := \hat{S}(\mathfrak{g}^*)\sharp U(\mathfrak{g}^R)$. Note that the tensor factors in this smash product are ordered differently than in the definition of H^L. The generators of H^R are $\hat{y}_\mu, \partial^\mu$, $\mu = 1, \ldots, n$. In addition to the relations in $U(\mathfrak{g}^R)$ and $\hat{S}(\mathfrak{g}^*)$, we also have

$$[\partial^\mu, \hat{y}_\nu] = \left(\frac{\mathcal{C}}{e^{\mathcal{C}}-1}\right)^\mu_\nu .$$

The corresponding realization, $\hat{y}_\nu \mapsto x_\lambda \tilde{\phi}^\lambda_\nu$, $\partial^\mu \mapsto \partial^\mu$, defines an isomorphism $H^R \cong \hat{A}_n$.

THEOREM 1. *There is an algebra isomorphism from $H^L = U(\mathfrak{g}^L)\sharp\hat{S}(\mathfrak{g}^*)$ to $H^R = \hat{S}(\mathfrak{g}^*)\sharp U(\mathfrak{g}^R)$ which fixes the commutative subalgebra $\hat{S}(\mathfrak{g}^*)$ (i.e. identifies $1\sharp\hat{S}(\mathfrak{g}^*)$ with $\hat{S}(\mathfrak{g}^*)\sharp 1$, and in particular $\partial^\mu \mapsto \partial^\mu$), and which maps $\hat{x}_\nu \mapsto \hat{y}_\sigma \mathcal{O}^\sigma_\nu$, where $\mathcal{O} := e^{\mathcal{C}}$ is an invertible $n\times n$-matrix with entries $\mathcal{O}^\mu_\nu \in \hat{S}(\mathfrak{g}^*)$ and inverse $\mathcal{O}^{-1} = e^{-\mathcal{C}}$. After the identification, $[\hat{x}_\mu, \hat{y}_\nu] = 0$. Consequently, the images of $U(\mathfrak{g}^L) \hookrightarrow H^L$ and $U(\mathfrak{g}^R) \hookrightarrow H^R$ mutually commute. The following identities hold*

$$[\mathcal{O}^\lambda_\mu, \hat{y}_\nu] = C^\lambda_{\rho\nu}\mathcal{O}^\rho_\mu \tag{9}$$

$$[\mathcal{O}^\lambda_\mu, \hat{x}_\nu] = C^\rho_{\mu\nu}\mathcal{O}^\lambda_\rho \tag{10}$$

$$[(\mathcal{O}^{-1})^\lambda_\mu, \hat{x}_\nu] = -C^\lambda_{\rho\nu}(\mathcal{O}^{-1})^\rho_\mu \tag{11}$$

$$[(\mathcal{O}^{-1})^\lambda_\mu, \hat{y}_\nu] = -C^\rho_{\mu\nu}(\mathcal{O}^{-1})^\lambda_\rho \tag{12}$$

$$C^\tau_{\mu\nu}\mathcal{O}^\lambda_\tau = C^\lambda_{\rho\sigma}\mathcal{O}^\rho_\mu\mathcal{O}^\sigma_\nu, \quad C^\tau_{\mu\nu}(\mathcal{O}^{-1})^\lambda_\tau = C^\lambda_{\rho\sigma}(\mathcal{O}^{-1})^\rho_\mu(\mathcal{O}^{-1})^\sigma_\nu. \tag{13}$$

Proof. The composition of the isomorphisms of algebras $H^L \cong \hat{A}_n$ and $\hat{A}_n \cong H^R$ is an isomorphism $H^L \cong H^R$. If we express $\hat{x}_\mu$ and $\hat{y}_\nu$ within $\hat{A}_n$ as $x_\rho\phi^\rho_\mu$ and $x_\sigma\tilde{\phi}^\sigma_\nu$ respectively, the commutation relation $[\hat{x}_\mu, \hat{y}_\nu] = 0$ becomes $[x_\rho\phi^\rho_\mu, x_\sigma\tilde{\phi}^\sigma_\nu] = 0$, which is the Proposition 5 (Appendix A.1). Comparing the formulas for ϕ and $\tilde{\phi}$, note that

$$\tilde{\phi} = \phi\, e^{-\mathcal{C}}, \qquad \hat{x}_\nu = \hat{y}_\mu (e^{\mathcal{C}})^\mu_\nu = \hat{y}_\mu\mathcal{O}^\mu_\nu. \tag{14}$$

Rewrite $[\hat{x}_\mu, \hat{y}_\nu]$ now as

$$[\hat{y}_\rho\mathcal{O}^\rho_\mu, \hat{y}_\nu] = [\hat{y}_\rho, \hat{y}_\nu]\mathcal{O}^\rho_\mu + \hat{y}_\lambda[\mathcal{O}^\lambda_\mu, \hat{y}_\nu] = \hat{y}_\lambda(-C^\lambda_{\rho\nu}\mathcal{O}^\rho_\mu + [\mathcal{O}^\lambda_\mu, \hat{y}_\nu]).$$

Starting with the evident fact $[\partial^\gamma, \hat{y}_\nu] \in \hat{S}(\mathfrak{g}^*)$, and using the induction, one shows $[\hat{S}(\mathfrak{g}^*), \hat{y}_\nu] \subset \hat{S}(\mathfrak{g}^*)$. Thus, $(-C^\lambda_{\rho\nu}\mathcal{O}^\rho_\mu + [\mathcal{O}^\lambda_\mu, \hat{y}_\nu]) \in \hat{S}(\mathfrak{g}^*)$. Elements $\hat{y}_\lambda$ are independent in H^R, which is here considered a right $\hat{S}(\mathfrak{g}^*)$-module, hence $0 = \hat{y}_\lambda(-C^\lambda_{\rho\nu}\mathcal{O}^\rho_\mu + [\mathcal{O}^\lambda_\mu, \hat{y}_\nu])$ implies (9). Similarly, in $[\hat{x}_\mu, \hat{y}_\nu] = 0$ replace $\hat{y}_\nu$ with $\hat{x}_\lambda(\mathcal{O}^{-1})^\lambda_\nu$ to prove (11). To show (10), calculate $C^\lambda_{\mu\nu}\hat{y}_\rho\mathcal{O}^\rho_\lambda = C^\lambda_{\mu\nu}\hat{x}_\lambda = [\hat{x}_\mu, \hat{x}_\nu] = [\hat{y}_\rho\mathcal{O}^\rho_\mu, \hat{x}_\nu] = \hat{y}_\rho[\mathcal{O}^\rho_\mu, \hat{x}_\nu]$, hence $\hat{y}_\rho(C^\lambda_{\mu\nu}\mathcal{O}^\rho_\lambda - [\mathcal{O}^\rho_\mu, \hat{x}_\nu]) = 0$. For (12) we reason analogously with $[\hat{x}_\rho(\mathcal{O}^{-1})^\rho_\mu, \hat{y}_\nu]$. If in (9) and (11) we replace $\hat{y}_\nu$ (resp. $\hat{x}_\nu$) on the left by $\hat{y}_\rho(\mathcal{O}^{-1})^\rho_\nu$ (resp. $\hat{x}_\rho\mathcal{O}^\rho_\nu$), we get a quadratic (in $\mathcal{O}$ or $\mathcal{O}^{-1}$) expression on the right, which is then compared with (10) and (12) to obtain (13).

3. Actions $\blacktriangleright$ and $\blacktriangleleft$ and some identities for them

There is a map $\epsilon_S : \hat{S}(\mathfrak{g}^*) \to \boldsymbol{k}$, taking a formal power series to its constant term ('evaluation at 0'). We introduce the 'black action' $\blacktriangleright$ of H^L on $U(\mathfrak{g}^L)$ as the composition

$$H^L \otimes U(\mathfrak{g}^L) \hookrightarrow H^L \otimes H^L \xrightarrow{m} H^L \cong U(\mathfrak{g}^L)\sharp\hat{S}(\mathfrak{g}^*) \xrightarrow{\mathrm{id}\sharp\epsilon_S} U(\mathfrak{g}^L), \quad (15)$$

where m is the multiplication map. $\blacktriangleright$ is the unique action for which $\partial^\mu \blacktriangleright 1 = 0$ for all μ and $\hat{f} \blacktriangleright 1 = \hat{f}$ for all $\hat{f} \in U(\mathfrak{g}^L)$. It follows that $\mathcal{O}^\mu_\nu \blacktriangleright 1 = \delta^\mu_\nu 1 = (\mathcal{O}^{-1})^\mu_\nu \blacktriangleright 1$ and $\hat{y}_\nu \blacktriangleright 1 = \hat{x}_\mu(\mathcal{O}^{-1})^\mu_\nu \blacktriangleright 1 = \delta^\mu_\nu \hat{x}_\mu = \hat{x}_\nu$. Similarly, the right black action $\blacktriangleleft$ of H^R on $U(\mathfrak{g}^R)$ is the composition

$$U(\mathfrak{g}^R) \otimes H^R \hookrightarrow H^R \otimes H^R \xrightarrow{m} H^R \cong \hat{S}(\mathfrak{g}^*)\sharp U(\mathfrak{g}^R) \xrightarrow{\epsilon_S\sharp\mathrm{id}} U(\mathfrak{g}^R),$$

characterized by $1 \blacktriangleleft \partial^\mu = 0$, and $1 \blacktriangleleft \hat{u} = \hat{u}$, for all $\hat{u} \in U(\mathfrak{g}^R)$.

THEOREM 2. *For any $\hat{f}, \hat{g} \in U(\mathfrak{g}^L)$ the following identities hold*

$$\hat{x}_\alpha \hat{f} = (\mathcal{O}^\beta_\alpha \blacktriangleright \hat{f})\hat{x}_\beta \quad (16)$$

$$\mathcal{O}^\gamma_\alpha \blacktriangleright (\hat{g}\hat{f}) = (\mathcal{O}^\beta_\alpha \blacktriangleright \hat{g})(\mathcal{O}^\gamma_\beta \blacktriangleright \hat{f}) \quad (17)$$

$$(\mathcal{O}^{-1})^\gamma_\alpha \blacktriangleright (\hat{g}\hat{f}) = ((\mathcal{O}^{-1})^\gamma_\beta \blacktriangleright \hat{g})((\mathcal{O}^{-1})^\beta_\alpha \blacktriangleright \hat{f}) \quad (18)$$

$$\hat{y}_\alpha \blacktriangleright \hat{f} = \hat{f}\hat{x}_\alpha \quad (19)$$

$$(\hat{x}_\alpha \blacktriangleright \hat{f})\hat{g} = (\mathcal{O}^\beta_\alpha \blacktriangleright \hat{f})(\hat{x}_\beta \blacktriangleright \hat{g}) \quad (20)$$

Proof. We show (16) for monomials $\hat{f}$ by induction on the degree of monomial; by linearity this is sufficient. For the base of induction, it is sufficient to note $\mathcal{O}^\beta_\alpha \blacktriangleright 1 = \delta^\beta_\alpha$. For the step of induction, calculate for arbitrary $\hat{f}$ of degree k

$$\begin{array}{rcl} \mathcal{O}^\gamma_\alpha \blacktriangleright (\hat{x}_\nu \hat{f}) & = & [\mathcal{O}^\gamma_\alpha, \hat{x}_\nu \hat{f}] \blacktriangleright 1 + \hat{x}_\nu \hat{f} \mathcal{O}^\gamma_\alpha \blacktriangleright 1 \\ & = & [\mathcal{O}^\gamma_\alpha, \hat{x}_\nu] \blacktriangleright \hat{f} + \hat{x}_\nu [\mathcal{O}^\gamma_\alpha, \hat{f}] \blacktriangleright 1 + \hat{x}_\nu \hat{f} \delta^\gamma_\alpha \\ & = & C^\beta_{\alpha\nu} \mathcal{O}^\gamma_\beta \blacktriangleright \hat{f} + \hat{x}_\nu (\mathcal{O}^\gamma_\alpha \blacktriangleright \hat{f}) \\ & = & (C^\beta_{\alpha\nu} + \delta^\beta_\alpha \hat{x}_\nu)(\mathcal{O}^\gamma_\beta \blacktriangleright \hat{f}) \\ & = & (\mathcal{O}^\beta_\alpha \blacktriangleright \hat{x}_\nu)(\mathcal{O}^\gamma_\beta \blacktriangleright \hat{f}), \end{array}$$

and use this result in the following:

$$\begin{array}{rcl} \hat{x}_\alpha \hat{x}_\nu \hat{f} & = & (\mathcal{O}^\beta_\alpha \blacktriangleright \hat{x}_\nu)\hat{x}_\beta \hat{f} \\ & = & (\mathcal{O}^\beta_\alpha \blacktriangleright \hat{x}_\nu)(\mathcal{O}^\gamma_\beta \blacktriangleright \hat{f})\hat{x}_\gamma \\ & = & (\mathcal{O}^\gamma_\alpha \blacktriangleright (\hat{x}_\nu \hat{f}))\hat{x}_\gamma. \end{array}$$

Thus (16) holds for $\hat{f}$-s of degree $k+1$, hence, by induction, for all. Along the way, we have also shown (17) for $\hat{g}$ of degree 1 and $\hat{f}$ arbitrary. Now we do induction on the degree of $\hat{g}$: replace $\hat{g}$ with $\hat{x}_\mu\hat{g}$ and calculate

$$\begin{aligned}\mathcal{O}^\gamma_\alpha \blacktriangleright ((\hat{x}_\mu\hat{g})\hat{f}) &= (\mathcal{O}^\beta_\alpha \blacktriangleright \hat{x}_\mu)(\mathcal{O}^\gamma_\beta \blacktriangleright (\hat{g}\hat{f}))\\ &= (\mathcal{O}^\beta_\alpha \blacktriangleright \hat{x}_\mu)(\mathcal{O}^\sigma_\beta \blacktriangleright \hat{g})(\mathcal{O}^\gamma_\sigma \blacktriangleright \hat{f})\\ &= (\mathcal{O}^\sigma_\alpha \blacktriangleright (\hat{x}_\mu\hat{g}))(\mathcal{O}^\gamma_\sigma \blacktriangleright \hat{f}).\end{aligned}$$

The proof of (18) is similar to (17) and left to the reader. To show (19), we use (16) and expression $\hat{y}_\alpha = \hat{x}_\beta(\mathcal{O}^{-1})^\beta_\alpha$:

$$\hat{x}_\beta(\mathcal{O}^{-1})^\beta_\alpha \blacktriangleright \hat{f} = \hat{x}_\beta \blacktriangleright ((\mathcal{O}^{-1})^\beta_\alpha \blacktriangleright \hat{f}) = (\mathcal{O}^\gamma_\beta \blacktriangleright ((\mathcal{O}^{-1})^\beta_\alpha \blacktriangleright \hat{f}))\hat{x}_\gamma$$

$$= ((\mathcal{O}^\gamma_\beta(\mathcal{O}^{-1})^\beta_\alpha) \blacktriangleright \hat{f})\hat{x}_\gamma = \delta^\gamma_\alpha\hat{f}\hat{x}_\gamma = \hat{f}\hat{x}_\alpha$$

Finally, (20) follows from (16) by multiplying from the right with $\hat{g}$, and using $\hat{x}_\beta \blacktriangleright \hat{g} = \hat{x}_\beta\hat{g}$ and $\hat{x}_\alpha \blacktriangleright \hat{f} - \hat{x}_\alpha\hat{f}$.

Now we state an analogue of the Theorem 2 for $\blacktriangleleft$.

THEOREM 3. *For any $\hat{f}, \hat{g} \in U(\mathfrak{g}^R)$ the following identities hold*

$$\hat{f}\hat{y}_\alpha = \hat{y}_\beta(\hat{f} \blacktriangleleft (\mathcal{O}^{-1})^\beta_\alpha), \tag{21}$$

$$(\hat{g}\hat{f}) \blacktriangleleft \mathcal{O}^\gamma_\alpha = (\hat{g} \blacktriangleleft \mathcal{O}^\gamma_\beta)(\hat{f} \blacktriangleleft \mathcal{O}^\beta_\alpha), \tag{22}$$

$$(\hat{g}\hat{f}) \blacktriangleleft (\mathcal{O}^{-1})^\gamma_\alpha = (\hat{g} \blacktriangleleft (\mathcal{O}^{-1})^\beta_\alpha)(\hat{f} \blacktriangleleft (\mathcal{O}^{-1})^\gamma_\beta) \tag{23}$$

$$\hat{f} \blacktriangleleft \hat{z}_\alpha = \hat{y}_\alpha\hat{f}, \tag{24}$$

$$\hat{g}(\hat{f} \blacktriangleleft \hat{y}_\alpha) = (\hat{g} \blacktriangleleft \hat{y}_\beta)(\hat{f} \blacktriangleleft (\mathcal{O}^{-1})^\beta_\alpha), \tag{25}$$

where

$$\hat{z}_\alpha := \mathcal{O}^\beta_\alpha\hat{y}_\beta = \mathcal{O}^\beta_\alpha\hat{x}_\rho(\mathcal{O}^{-1})^\rho_\beta \in H^L \cong H^R. \tag{26}$$

To emphasize the special role of the matrix $\mathcal{O}$, we make the following remark [18]. The topological Hopf algebra $\hat{S}(\mathfrak{g}^{L*})$ coacts on $U(\mathfrak{g}^L)$ by a unique right coaction $\rho^{\mathrm{YD}} : U(\mathfrak{g}^L) \to U(\mathfrak{g}^L) \otimes \hat{S}(\mathfrak{g}^{L*})$, which is an algebra *anti*homomorphism and which is on generators $\hat{x}_\mu \in \mathfrak{g}^L$ given by

$$\rho^{\mathrm{YD}} : \hat{x}_\mu \mapsto \hat{x}_\rho \otimes (\mathcal{O}^{-1})^\rho_\mu.$$

Postcomposing this coaction with the inclusion $U(\mathfrak{g}^L) \otimes \hat{S}(\mathfrak{g}^{L*}) \hookrightarrow U(\mathfrak{g}^L)\hat{\otimes}\hat{S}(\mathfrak{g}^{L*})$ makes $U(\mathfrak{g}^L)$ into a braided-commutative algebra in the category of Yetter-Drinfeld modules over $\hat{S}(\mathfrak{g}^{L*})$, internally in a monoidal category of complete cofiltered vector spaces (see Appendix

A.2 and [18]), which places our Hopf algebroid below into a version of the scalar extension Hopf algebroid [2, 6].

4. Completed tensor product and bimodules

In this section, we discuss the completed tensor products needed for the coproducts ($\Delta_{S(\mathfrak{g}^*)}$ in this and Δ^L and Δ^R in the next section), and introduce the maps $\alpha^L, \beta^L, \alpha^R, \beta^R$ and use them to define $U(\mathfrak{g}^L)$-bimodule structure on H^L and $U(\mathfrak{g}^R)$-bimodule structure on H^R.

The inclusions of filtered components $U_k(\mathfrak{g}) \subset U_{k+1}(\mathfrak{g}) \subset U(\mathfrak{g})$ induce epimorphisms of dual vector spaces $U(\mathfrak{g})^* \to U_{k+1}(\mathfrak{g})^* \to U_k(\mathfrak{g})^*$, hence a complete *cofiltration* on $U(\mathfrak{g})^* = \varprojlim_k U_k(\mathfrak{g})^*$ (see Appendix A.2). For each finite level k, $U_k(\mathfrak{g})$ is finite dimensional, hence $(U_k(\mathfrak{g}) \otimes U_l(\mathfrak{g}))^* \cong U_k(\mathfrak{g})^* \otimes U_l(\mathfrak{g})^*$. Thus the multiplication $U_k(\mathfrak{g}) \otimes U_l(\mathfrak{g}) \to U_{k+l}(\mathfrak{g}) \subset U(\mathfrak{g})$ dualizes to $\Delta_{k,l} : U(\mathfrak{g})^* \to U_k(\mathfrak{g})^* \otimes U_l(\mathfrak{g})^*$. The inverse limits $\varprojlim_k \Delta_{k,k}$ and $\varprojlim_p \varprojlim_q \Delta_{p,q}$ agree and define the coproduct $\Delta_{U(\mathfrak{g})^*} := \varprojlim_k \Delta_{k,k} : U(\mathfrak{g})^* \to \varprojlim_k U_k(\mathfrak{g})^* \otimes U_k(\mathfrak{g})^* \cong \varprojlim_p \varprojlim_q U_p(\mathfrak{g})^* \otimes U_q(\mathfrak{g})^*$. The right-hand side is by definition the completed tensor product, $U(\mathfrak{g})^* \hat{\otimes} U(\mathfrak{g})^*$. (For completed tensoring of *elements* and *maps* we below often use simplified notation, $\otimes$.) Coproduct $\Delta_{U(\mathfrak{g})^*}$ transfers, along the isomorphism $\xi^T : U(\mathfrak{g})^* \xrightarrow{\cong} S(\mathfrak{g})^*$ of cofiltered algebras (see (3)), to the topological coproduct on the completed symmetric algebra $\hat{S}(\mathfrak{g}^*) \cong S(\mathfrak{g})^*$ (cf. [19]),

$$\Delta_{\hat{S}(\mathfrak{g}^*)} : \hat{S}(\mathfrak{g}^*) \to \hat{S}(\mathfrak{g}^*) \hat{\otimes} \hat{S}(\mathfrak{g}^*).$$

This construction can be performed both for $\mathfrak{g}^L$ and $\mathfrak{g}^R$. The canonical isomorphism of Hopf algebras $U(\mathfrak{g}^R) \cong U(\mathfrak{g}^L)^{\mathrm{op}}$ induces the isomorphism of dual cofiltered Hopf algebras $U(\mathfrak{g}^R)^* \cong (U(\mathfrak{g}^L)^*)^{\mathrm{co}}$, commuting with ξ^T, hence inducing an isomorphism of Hopf algebras $\hat{S}(\mathfrak{g}^{R*}) \cong \hat{S}(\mathfrak{g}^{L*})^{\mathrm{co}}$ fixing the underlying algebra $\hat{S}(\mathfrak{g}^*)$. Thus, the coproduct on $\hat{S}(\mathfrak{g}^{R*})$ is $\Delta^{\mathrm{op}}_{\hat{S}(\mathfrak{g}^{L*})}$, hence we just write $\hat{S}(\mathfrak{g}^*)$ and use the algebra identification, with the (co)opposite signs $\hat{S}(\mathfrak{g}^*)^{\mathrm{co}}$ or $\Delta^{\mathrm{op}}_{\hat{S}(\mathfrak{g}^*)}$ when needed.

As discussed in [19, 20], the coproduct is equivalently characterized by

$$P \blacktriangleright (\hat{f}\hat{g}) = m(\Delta_{\hat{S}(\mathfrak{g}^*)}(P)(\blacktriangleright \otimes \blacktriangleright)(\hat{f} \otimes \hat{g})), \tag{27}$$

for all $P \in \hat{S}(\mathfrak{g}^*)$ (for instance, $P = \partial^\mu$) and all $\hat{f}, \hat{g} \in U(\mathfrak{g})$. Using the action $\blacktriangleright$ we assumed that we embedded $\hat{S}(\mathfrak{g}^*) \hookrightarrow H^R \cong \hat{A}_n$. The right hand version of (27) is that for all $\hat{u}, \hat{v} \in U(\mathfrak{g}^R)$ and $Q \in \hat{S}(\mathfrak{g}^*)$,

$$(\hat{u}\hat{v}) \blacktriangleleft Q = m((\hat{u} \otimes \hat{v})(\blacktriangleleft \otimes \blacktriangleleft)\Delta^{\mathrm{op}}_{\hat{S}(\mathfrak{g}^*)}(Q)). \tag{28}$$

DEFINITION 2. *The homomorphism $\alpha^L : U(\mathfrak{g}^L) \hookrightarrow H^L$ is the inclusion $U(\mathfrak{g}^L) \to U(\mathfrak{g}^L)\sharp 1 \hookrightarrow U(\mathfrak{g}^L)\sharp\hat{S}(\mathfrak{g}^*) = H^L$ and $\alpha^R : U(\mathfrak{g}^R) \to H^R$ is the inclusion $\alpha^R : U(\mathfrak{g}^R) \to 1\sharp U(\mathfrak{g}^R) \hookrightarrow \hat{S}(\mathfrak{g}^*)\sharp U(\mathfrak{g}^R) = H^R$. Thus, in our writing conventions, $\alpha^L(\hat{f}) = \hat{f}$ and $\alpha^R(\hat{u}) = \hat{u}$. Likewise, $\beta^L : U(\mathfrak{g}^L)^{\mathrm{op}} \to H^L$ and $\beta^R : U(\mathfrak{g}^R)^{\mathrm{op}} \to H^R$ are the unique antihomomorphisms of algebras extending the formulas (cf. (26))*

$$\begin{array}{l} \beta^L(\hat{x}_\mu) = \hat{x}_\rho(\mathcal{O}^{-1})^\rho_\mu = \hat{y}_\mu \in H^L. \\ \beta^R(\hat{y}_\alpha) := \mathcal{O}^\rho_\alpha \hat{y}_\rho = \mathcal{O}^\rho_\alpha \hat{x}_\sigma (\mathcal{O}^{-1})^\sigma_\rho = \hat{z}_\alpha \in H^R. \end{array} \tag{29}$$

The extension β^L exists, because the extension of the map $\hat{x}_\mu \mapsto \hat{y}_\mu$ on $\mathfrak{g}$ to the antihomomorphism $\beta^L_{T(\mathfrak{g})} : T(\mathfrak{g}) \to H^L$ maps $[\hat{x}_\alpha, \hat{x}_\beta] - C^\gamma_{\alpha\beta}\hat{x}_\gamma$ to $[\hat{y}_\beta, \hat{y}_\alpha] - C^\gamma_{\alpha\beta}\hat{y}_\gamma = 0$; similarly for β^R.

PROPOSITION 1. *(i) H^L is a $U(\mathfrak{g}^L)$-bimodule via the formula $a.h.b := \alpha^L(a)\beta^L(b)h$, for all $a, b \in U(\mathfrak{g}^L)$, $h \in H^L$. Likewise, H^R is a $U(\mathfrak{g}^R)$-bimodule via $a.h.b := h\beta^R(a)\alpha^R(b)$, for all $a, b \in U(\mathfrak{g}^R)$, $h \in H^R$.* From now on these bimodule structures are assumed.

(ii) For any $\hat{f}, \hat{g} \in U(\mathfrak{g}^L)$ and any $\hat{u}, \hat{v} \in U(\mathfrak{g}^R)$,

$$\beta^L(\hat{g}) \blacktriangleright \hat{f} = \hat{f}\hat{g}, \quad \hat{u} \blacktriangleleft \beta^R(\hat{v}) = \hat{v}\hat{u}. \tag{30}$$

Proof. (i) The bimodule property of commuting of the left and the right $U(\mathfrak{g}^L)$-action is ensured by $[\hat{x}_\mu, \hat{y}_\nu] = 0$. For the $U(\mathfrak{g}^R)$-actions it boils down to $[\hat{y}_\mu, \mathcal{O}^\rho_\nu \hat{x}_\sigma (\mathcal{O}^{-1})^\sigma_\rho] = 0$, which follows from the Theorem 1.

(ii) follows from (19) and (24), by induction on the filtered degree of $\hat{g}$ (respectively, of $\hat{v}$).

PROPOSITION 2. *Let $\hat{H}^L := U(\mathfrak{g}^L)\hat{\sharp}\hat{S}(\mathfrak{g}^*)$ and $\hat{H}^R := \hat{S}(\mathfrak{g}^*)\hat{\sharp}U(\mathfrak{g}^R)$ be the completed smash product cofiltered algebras defined in Theorem 6.*

(i) The inclusions $H^L\hat{\otimes}H^L \to \hat{H}^L\hat{\otimes}\hat{H}^L$, $H^R\hat{\otimes}H^R \to \hat{H}^R\hat{\otimes}\hat{H}^R$, $H^L\hat{\otimes}_{U(\mathfrak{g}^L)}H^L \to \hat{H}^L\hat{\otimes}_{U(\mathfrak{g}^L)}\hat{H}^L$ and $H^R\hat{\otimes}_{U(\mathfrak{g}^R)}H^R \to \hat{H}^R\hat{\otimes}_{U(\mathfrak{g}^R)}\hat{H}^R$ are isomorphisms of algebras;

(ii) The actions $\blacktriangleright$ and $\blacktriangleleft$ extend to the actions of the completed algebra $\blacktriangleright: \hat{H}^L \otimes U(\mathfrak{g}^L) \to U(\mathfrak{g}^L)$ and $\blacktriangleleft: U(\mathfrak{g}^R) \otimes \hat{H}^R \to U(\mathfrak{g}^R)$.

Proof. (i) $H^L_r = \hat{H}^L_r$, hence both sides of the tensor product inclusions have by the definition the same cofiltered components, hence they also have the same completion.

For (ii) extend the recipe from (15) and notice that $\mathrm{id}\hat{\sharp}\epsilon_S$ kills also all elements in $U(\mathfrak{g}^L)\hat{\sharp}\hat{S}(\mathfrak{g}^*)$ not in $U(\mathfrak{g}^L)\sharp\hat{S}(\mathfrak{g}^*)$ with the result in $U(\mathfrak{g})$.

On the other hand, *there are no completed actions* $\hat{H}^L\hat{\otimes}U(\mathfrak{g}^L)\to U(\mathfrak{g}^L)$ and $U(\mathfrak{g}^R)\hat{\otimes}\hat{H}^R\to U(\mathfrak{g}^R)$ extending $\blacktriangleright$ and $\blacktriangleleft$.

DEFINITION 3. *The right ideal* $I\subset H^L\otimes H^L$ *is generated by the set of all elements of the form* $\beta^L(\hat{f})\otimes 1-1\otimes\alpha^L(\hat{f})$ *where* $\hat{f}\in H^L$. *In other words,* I *is the kernel of the canonical map* $H^L\otimes H^L\to H^L\otimes_{U(\mathfrak{g}^L)}H^L$.

The right ideal $I'\subset H^L\otimes H^L$ *is the set of all* $\sum_i h_i\otimes h'_i\in H^L\otimes H^L$ *such that*

$$\sum_{i,j}(h_i\blacktriangleright\hat{f}_j)(h'_i\blacktriangleright\hat{g}_j)=0,\quad\text{for all }\sum_j\hat{f}_j\otimes\hat{g}_j\in U(\mathfrak{g}^L)\otimes U(\mathfrak{g}^L).$$

Similarly, $\tilde{I}:=\ker(H^R\otimes H^R\to H^R\otimes_{U(\mathfrak{g}^R)}H^R)$ *is the left ideal in* $H^R\otimes H^R$ *generated by all elements of the form* $\alpha^R(\hat{u})\otimes 1-1\otimes\beta^R(\hat{u})$, $\hat{u}\in U(\mathfrak{g}^R)$, *and* $\tilde{I}'$ *is the left ideal in* $H^R\otimes H^R$ *consisting of all* $\sum_i h_i\otimes h'_i$ *such that* $\sum_{i,j}(\hat{u}_j\blacktriangleleft h_i)(\hat{v}_j\blacktriangleleft h'_i)=0$ *for all* $\sum_j\hat{u}_j\otimes\hat{v}_j\in U(\mathfrak{g}^R)\otimes U(\mathfrak{g}^R)$. *The completions (Appendix A.2) of ideals* I,I' *and* $\tilde{I},\tilde{I}'$ *are denoted* $\hat{I},\hat{I}'\subset H^L\hat{\otimes}H^L\cong\hat{H}^L\hat{\otimes}\hat{H}^L$ *and* $\hat{\tilde{I}},\hat{\tilde{I}}'\subset H^R\hat{\otimes}H^R\cong\hat{H}^R\hat{\otimes}\hat{H}^R$, *respectively.*

More generally, for $r\geq 2$, let $I^{(r)}$ be the kernel of the canonical projection $(H^L)^{\otimes r}:=H^L\otimes H^L\otimes\ldots\otimes H^L$ (r factors) to the tensor product of $U(\mathfrak{g}^L)$-bimodules $H^L\otimes_{U(\mathfrak{g}^L)}H^L\otimes_{U(\mathfrak{g}^L)}\ldots\otimes_{U(\mathfrak{g}^L)}H^L$. $I^{(r)}$ coincides with the smallest right ideal in the tensor product algebra $(H^L)^{\otimes r}$ which contains $1^{\otimes k}\otimes I\otimes 1^{\otimes(r-k-2)}$ for $k=0,\ldots,r-2$. Let $I'^{(r)}$ be the set of all elements $\sum_i h_{1i}\otimes h_{2i}\otimes\ldots h_{ri}\in(H^L)^{\hat{\otimes}r}$ such that for every $\sum_j u_{1j}\otimes u_{2j}\otimes\ldots\otimes u_{rj}\in U(\mathfrak{g}^L)^{\otimes r}$

$$\sum_{i,j}(h_{1i}\blacktriangleright u_{1j})(h_{2i}\blacktriangleright u_{2j})\cdots(h_{ri}\blacktriangleright u_{rj})=0.$$

LEMMA 1. *(i) There is a nondegenerate (topological) Hopf pairing* $\langle,\rangle_\phi:U(\mathfrak{g})\hat{\otimes}\hat{S}(\mathfrak{g}^*)\to\boldsymbol{k}$ *given by* $\langle\hat{u},P\rangle_\phi:=\boldsymbol{\phi}(\hat{u})(P)(1)$ *where the action on 1 is the Fock action (i.e. evaluating* $\epsilon_{\hat{S}(\mathfrak{g}^*)}$*). It satisfies the* Heisenberg double identity $P\blacktriangleright\hat{u}=\sum\langle\hat{u}_{(2)},P\rangle\hat{u}_{(1)}$.

(ii) For multiindices J_1,J_2,J *such that* $J_1+J_2=J$,

$$\boldsymbol{\phi}(\hat{x}_I)(\partial^J)=\sum_{I_1+I_2=I}\frac{I!}{I_1!I_2!}\boldsymbol{\phi}(\hat{x}_{I_1})(\partial^{J_1})\boldsymbol{\phi}(\hat{x}_{I_2})(\partial^{J_2}).$$

(iii) $\boldsymbol{\phi}(\hat{x}_I)(\partial^J)\in\hat{S}(\mathfrak{g}^*)_{|J|-|I|}$ *if* $|I|<|J|$.

(iv) $\boldsymbol{\phi}(\hat{x}_I)(\partial^J)-I!\,\delta^I_J\in\hat{S}(\mathfrak{g}^*)_1$ *if* $|I|=|J|$.

(v) For multiindices K, J and for the basis $\{\partial^K \in S(\mathfrak{g}^*)\}_K$ the identities $\langle \partial^K, \hat{x}_J\rangle = K!\,\delta^K_J$ hold if $K \geq J$ (in partial order for multiindices), but in general not otherwise.

(vi) There is a unique family $\{\partial^{\{K\}} \in \hat{S}(\mathfrak{g}^*)\}_K$ which for <u>all</u> multiindices K, J satisfies $\langle \partial^{\{K\}}, \hat{x}_J\rangle_\phi = K!\,\delta^K_J$.

(vii) Let $f \in \hat{S}(\mathfrak{g}^*)$. Then $\forall r \in \mathbb{N}_0$, $f_r = \sum_J \frac{1}{J!}\langle f, \hat{x}_J\rangle_\phi \partial^{\{J\}}_r \in S(\mathfrak{g})_r$, where the sum is finite because $\partial^{\{J\}}_r = 0$ if $r < |J|$. Thus, there is a formal sum representation $f = \varprojlim_r f_r = \sum_J \frac{1}{J!}\langle f, \hat{x}_J\rangle_\phi \partial^{\{J\}}$.

(viii) $\partial^J = \sum_{|K|\geq|J|} d_{K,J}\partial^{\{K\}}$ for some $d_{K,J} \in \boldsymbol{k}$.

Proof. (i) is a part of the content of Theorems 3.3 and 3.5 in [20].

(ii) The action is Hopf, hence the identity follows from the formula $\Delta(\hat{x}_I) = \sum_{I_1+I_2=I} \frac{I!}{I_1!I_2!}\hat{x}_{I_1} \otimes \hat{x}_{I_2}$ for the coproduct in $U(\mathfrak{g})$.

(iii) This is simple induction on $|J| - |I|$ using (ii) and $\boldsymbol{\phi}(1)(\partial^K) = \partial^K \in \hat{S}(\mathfrak{g}^*)$.

(iv) follows by induction on $|I|$ using (ii), (iii) and $\boldsymbol{\phi}(\hat{x}_\mu)(\partial^\mu) = \phi^\nu_\mu$, which by (6) equals δ^ν_μ up to a summand in $\hat{S}(\mathfrak{g}^*)_1$.

(v) This is an application of the formula for $\langle,\rangle_\phi$ in (i) to the results (iii) and (iv); indeed the elements in $\hat{S}(\mathfrak{g}^*)_1$ vanish when applied to 1.

(vi) Denote, as in Appendix A.2, by $\pi_r : \hat{S}(\mathfrak{g}^*) \to S(\mathfrak{g}^*)_r$ and $\pi_{r,r+s} : S(\mathfrak{g}^*)_{r+s} \to S(\mathfrak{g}^*)_r$ the canonical projections. By [20], 3.4, the isomorphism $\xi^T : U(\mathfrak{g})^* \to \hat{S}(\mathfrak{g}^*)$ (see (3)) of cofiltered algebras identifies the pairing $\langle,\rangle_\phi$ with the evaluation pairing $\langle,\rangle_U$ between $U(\mathfrak{g})^*$ and $U(\mathfrak{g})$. By the properties of $\langle,\rangle_U$, for each $r \in \mathbb{N}_0$, the induced pairing $\langle,\rangle_r : \hat{S}(\mathfrak{g}^*)_r \otimes U(\mathfrak{g})_r \to \boldsymbol{k}$ characterized by $\langle \pi_r(P), \hat{u}\rangle_r = \langle P, \hat{u}\rangle_\phi$ for each $P \in \hat{S}(\mathfrak{g}^*)$, $\hat{u} \in U(\mathfrak{g})$ is *nondegenerate*. Thus there is a basis $\{\partial^{\{K\}}_r\}_{|K|\leq r}$ dual to $\{\hat{x}_L\}_{|L|\leq r}$. Now $\ker \pi_{r,r+s} = \mathrm{Span}\,\{\partial^J,\, r < |J| \leq r+s\}$. By (v) $\langle \ker \pi_{r,r+s}, U(\mathfrak{g})_r\rangle_{r+s} = 0$. Therefore for all K, L, $|K| \leq r$, $|L| \leq r$, $\delta^K_L = \langle \partial^{\{K\}}_{r+s}, \hat{x}_L\rangle_{r+s} = \langle \pi_{r,r+s}(\partial^{\{K\}}_{r+s}), \hat{x}_L\rangle_r = \langle \partial^{\{K\}}_r, \hat{x}_L\rangle_r$. By nondegeneracy, $\pi_{r,r+s}(\partial^{\{K\}}_{r+s}) = \partial^{\{K\}}_r$. Therefore $\exists!\,\partial^{\{K\}} \in \hat{S}(\mathfrak{g}^*)_{r+s}$ such that $\pi_r(\partial^{\{K\}}) = \partial^{\{K\}}_r$ for $r \geq |K|$ and $\pi_r(\partial^{\{K\}}) = 0$ for $r < |K|$; the requirements of (vi) hold for $\{\partial^{\{K\}}\}_K$.

(vii) is now straightforward and (viii) follows from (v) and (vii).

THEOREM 4. *(i) The restriction of $\blacktriangleright$: $H^L \otimes U(\mathfrak{g}^L) \to U(\mathfrak{g}^L)$ to $\hat{S}(\mathfrak{g}^*) \otimes U(\mathfrak{g}^L) \to U(\mathfrak{g}^L)$ turns $U(\mathfrak{g}^L)$ into a* **faithful** *left $\hat{S}(\mathfrak{g}^*)$-module.*

(ii) The right ideals I, I' agree and the left ideals $\tilde{I}, \tilde{I}'$ agree.

(iii) More generally, $I^{(r)} = I'^{(r)}$, $\tilde{I}^{(r)} = \tilde{I}'^{(r)}$ for $r \geq 2$.

(iv) Statements (ii) and (iii) hold also for the completed ideals.

Proof. We show part (ii) for the right ideals, $I = I'$; the method of the proof easily extends to the left ideals, and to (i), (iii) and (iv).

Let $\sum_\sigma \hat{f}_\sigma \otimes \hat{g}_\sigma \in I$ and $v = \hat{x}_{\mu_1} \cdots \hat{x}_{\mu_k}$ a monomial in $U(\mathfrak{g}^L)$. Then

$$(\beta^L(v) \blacktriangleright \hat{f}_\sigma)\hat{g}_\sigma - \hat{f}_\sigma \alpha^L(v) \blacktriangleright \hat{g}_\sigma = (\hat{y}_{\mu_k} \cdots \hat{y}_{\mu_1} \blacktriangleright \hat{f}_\sigma)\hat{g}_\sigma - \hat{f}_\sigma \hat{x}_{\mu_1} \cdots \hat{x}_{\mu_k} \blacktriangleright \hat{g}_\sigma,$$

which is zero by Eq. (19) and induction on k. Thus, by linearity, $I \subset I'$.

It remains to show the converse inclusion, $I' \subset I$. Suppose on the contrary that there is an element $\sum_\lambda h_\lambda \otimes h'_\lambda$ in I', but not in I; then after adding any element in I the sum is still in I' and not in I. Observe that $\hat{x}_J \partial^K \otimes \hat{x}_{J'} \partial^{K'} = \hat{x}_J \partial^K \otimes \alpha^L(\hat{x}_{J'})\partial^{K'} = \beta^L(\hat{x}_{J'})\hat{x}_J \partial^K \otimes \partial^{K'}$ mod I. The tensor factor $\beta(\hat{x}_{J'})\hat{x}_J\partial^K$ belongs to $H^L \subset \hat{H}^L$, hence it is also a formal linear combination of elements of the form $\hat{x}_{J''}\partial^{K''}$. Therefore, without loss of generality, we can assume

$$\sum_\lambda h_\lambda \otimes h'_\lambda = \sum_{J,K,L} a_{JKL} \hat{x}_J \partial^K \otimes \partial^L. \tag{31}$$

Using Lemma 1 (vi),(vii),(viii) we can in (31) uniquely express ∂^K as a formal sum in $\partial^{\{K\}}$ and ∂^L as a formal sum in $\partial^{\{L\}}$. Therefore, we can write $\sum_\lambda h_\lambda \otimes h'_\lambda$ as a formal sum

$$\sum_\lambda h_\lambda \otimes h'_\lambda = \sum_{J,K,L} b_{JKL} \hat{x}_J \partial^{\{K\}} \otimes \partial^{\{L\}},$$

for some coefficients $b_{JKL} \in \boldsymbol{k}$. The assumption $\sum_\lambda h_\lambda \otimes h'_\lambda \in I'$ implies

$$\sum_\lambda (h_\lambda \blacktriangleright \hat{x}_M)(h'_\lambda \blacktriangleright \hat{x}_N) = 0.$$

Choose multiindices M and N such that $(|M|, |N|)$ is a minimal bidegree for which b_{JMN} does not vanish for at least some J. By Lemma 1 (i), the formula $\Delta(\hat{x}_M) = \sum_{M_1+M_2=M} \frac{M!}{M_1! M_2!} \hat{x}_{M_1} \otimes \hat{x}_{M_2}$ for the coproduct in $U(\mathfrak{g})$, and Lemma 1 (vi)

$$\partial^{\{K\}} \blacktriangleright \hat{x}_M = \sum_{M_1+M_2=M} \binom{M}{M_2} \langle \partial^{\{K\}}, \hat{x}_{M_2} \rangle_\phi \, \hat{x}_{M_1} = \begin{cases} \binom{M}{K} \hat{x}_{M-K}, & M \geq K \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, using the minimality of $(|M|, |N|)$, only the summand with $M = K$ and $N = L$ contributes to the sum and

$$0 = \sum_\lambda (h_\lambda \blacktriangleright \hat{x}_M)(h'_\lambda \blacktriangleright \hat{x}_N) = \sum_J b_{JMN} \hat{x}_J,$$

hence by the linear independence of monomials $\hat{x}_J$, all $b_{JMN} = 0$, in the contradiction with the existence of J with b_{JMN} different from 0.

5. Bialgebroid structures

Let us now use the shorter notation $\mathcal{A}^L := U(\mathfrak{g}^L)$, $\mathcal{A}^R := U(\mathfrak{g}^R)$. A suggestive symbol $\mathcal{A}$ denotes an abstract algebra in the axioms where either $\mathcal{A}^L$ or $\mathcal{A}^R$ (or both) may substitute in here intended examples. In this section, we equip the isomorphic associative algebras H^L and H^R with different structures: H^L is a left $\mathcal{A}^L$-bialgebroid and H^R is a right $\mathcal{A}^R$-bialgebroid. We start by exhibiting the coring structures of these bialgebroids; an $\mathcal{A}$-coring is an analogue of a coalgebra where the ground field is replaced by a noncommutative algebra $\mathcal{A}$.

DEFINITION 4. *[2, 7] Let $\mathcal{A}$ be a unital algebra and C an $\mathcal{A}$-bimodule with left action $(a,c) \mapsto a.c$ and right action $(c,a) \mapsto c.a$. A triple (C,Δ,ϵ) is an $\mathcal{A}$-**coring** if*

(i) $\Delta : C \to C \otimes_{\mathcal{A}} C$ and $\epsilon : C \to \mathcal{A}$ are $\mathcal{A}$-bimodule maps; they are called the coproduct (comultiplication) and the counit;

(ii) Δ is coassociative: $(\Delta \otimes_{\mathcal{A}} \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes_{\mathcal{A}} \Delta) \circ \Delta$, where in the codomain the associativity isomorphism $(C \otimes_{\mathcal{A}} C) \otimes_{\mathcal{A}} C \cong C \otimes_{\mathcal{A}} (C \otimes_{\mathcal{A}} C)$ for the $\mathcal{A}$-bimodule tensor product is understood;

(iii) The counit axioms $(\epsilon \otimes_{\mathcal{A}} \mathrm{id}) \circ \Delta \cong \mathrm{id} \cong (\mathrm{id} \otimes_{\mathcal{A}} \epsilon) \circ \Delta$ hold, where the identifications of $\mathcal{A}$-bimodules $C \otimes_{\mathcal{A}} \mathcal{A} \cong C$, $c \otimes a \mapsto c.a$ and $\mathcal{A} \otimes_{\mathcal{A}} C \cong C$, $a \otimes d \mapsto a.d$ are understood.

PROPOSITION 3. *(i) $\exists!$ linear maps $\Delta^L : H^L \to H^L \hat{\otimes}_{\mathcal{A}^L} H^L$ and $\Delta^R : H^R \to H^R \hat{\otimes}_{\mathcal{A}^R} H^R$ such that Δ^L and Δ^R respectively satisfy*

$$P \blacktriangleright (\hat{f}\hat{g}) = m(\Delta^L(P)(\blacktriangleright \otimes \blacktriangleright)(\hat{f} \otimes \hat{g})), \quad \hat{f}, \hat{g} \in \mathcal{A}^L, \ P \in H^L, \tag{32}$$

$$(\hat{u}\hat{v}) \blacktriangleleft Q = m((\hat{u} \otimes \hat{v})(\blacktriangleleft \otimes \blacktriangleleft)\Delta^R(Q)), \quad \hat{u}, \hat{v} \in \mathcal{A}^R, \ Q \in H^R. \tag{33}$$

(ii) Δ^L is the unique left $\mathcal{A}^L$-module map $H^L \to H^L \hat{\otimes} H^L$ extending $\Delta_{\hat{S}(\mathfrak{g}^*)} : \hat{S}(\mathfrak{g}^*) \to \hat{S}(\mathfrak{g}^*) \hat{\otimes} \hat{S}(\mathfrak{g}^*) \subset H^L \hat{\otimes} H^L$. Likewise, Δ^R is the unique right $\mathcal{A}^R$-module map extending $\Delta^{\mathrm{op}}_{\hat{S}(\mathfrak{g}^*)}$ from $\hat{S}(\mathfrak{g}^*)$ to H^R. Equivalently,

$$\Delta^L(\hat{f} \sharp P) = \hat{f} \Delta_{\hat{S}(\mathfrak{g}^*)}(P), \qquad \Delta^R(Q \sharp \hat{v}) = \Delta^{\mathrm{op}}_{\hat{S}(\mathfrak{g}^*)}(Q)\hat{v}, \tag{34}$$

for all $P, Q \in \hat{S}(\mathfrak{g}^*)$, $\hat{f} \in \mathcal{A}^L$ and $\hat{v} \in \mathcal{A}^R$. In particular, $\Delta^L(\hat{x}_\mu) = \hat{x}_\mu \otimes 1$ and $\Delta^R(\hat{y}_\mu) = 1 \otimes \hat{y}_\mu$.

(iii) $\Delta^L(\mathcal{O}^\mu_\nu) = \mathcal{O}^\gamma_\nu \otimes \mathcal{O}^\mu_\gamma$, $\Delta^R(\mathcal{O}^\mu_\nu) = \mathcal{O}^\mu_\gamma \otimes \mathcal{O}^\gamma_\nu$,
$\Delta^L(\mathcal{O}^{-1})^\mu_\nu = (\mathcal{O}^{-1})^\mu_\gamma \otimes (\mathcal{O}^{-1})^\gamma_\nu$, $\Delta^R(\mathcal{O}^{-1})^\mu_\nu = (\mathcal{O}^{-1})^\gamma_\nu \otimes (\mathcal{O}^{-1})^\mu_\gamma$,
$\Delta^L(\hat{y}_\nu) = \Delta^L(\hat{x}_\mu(\mathcal{O}^{-1})^\mu_\nu) = \hat{x}_\mu(\mathcal{O}^{-1})^\mu_\gamma \otimes (\mathcal{O}^{-1})^\gamma_\nu = 1 \otimes \hat{y}_\nu$,
$\Delta^R(\hat{y}_\nu) = \Delta^R(\hat{y}_\mu \mathcal{O}^\mu_\nu) = 1 \otimes \hat{x}_\nu$.

(iv) $(H^L, \Delta^L, \epsilon^L)$ and $(H^R, \Delta^R, \epsilon^R)$ satisfy the axioms for $\mathcal{A}^L$-coring and $\mathcal{A}^R$-coring respectively, provided we replace the tensor product of

bimodules by the completed tensor of (cofiltered) bimodules and the counit modify as $(\epsilon\hat{\otimes}_{\mathcal{A}}\mathrm{id})\circ\Delta \cong j \cong (\mathrm{id}\hat{\otimes}_{\mathcal{A}}\epsilon)\circ\Delta$ *where, instead of the identity,* j *is the canonical map into the completion (say,* $j^L : H^L \hookrightarrow \hat{H}^L \cong H^L\hat{\otimes}\boldsymbol{k} \cong \boldsymbol{k}\hat{\otimes}H^L$*).*

Taking into account our bimodule structures, the counit axioms, Definition 4 (iii), read

$$\begin{array}{l} \sum \alpha^L(\epsilon^L(h_{(1)}))h_{(2)} = h = \sum \beta^L(\epsilon^L(h_{(2)}))h_{(1)}, \quad h \in H^L \\ \sum h_{(2)}\beta^R(\epsilon^R(h_{(1)})) = h = \sum h_{(1)}\alpha^R(\epsilon^R(h_{(2)})), \quad h \in H^R. \end{array} \tag{35}$$

(v) The coring structures from (iv) canonically extend to of an internal $\mathcal{A}^L$*-coring* $(\hat{H}^L, \hat{\Delta}^L, \hat{\epsilon}^L)$ *and an internal* $\mathcal{A}^R$*-coring* $(\hat{H}^R, \hat{\Delta}^R, \hat{\epsilon}^R)$ *(see [3]) in the category of complete cofiltered vector spaces with* $\hat{\otimes}$*-tensor product (see Proposition 2 and Appendix A.2). Bimodule structures on* $\hat{H}^L$*,* $\hat{H}^R$ *involve homomorphisms* $\hat{\alpha}^L := j^L\circ\alpha^L$*,* $\hat{\alpha}^R := j^R\circ\alpha^R$*, and antihomomorphisms* $\hat{\beta}^L := j^L\circ\beta^L$*,* $\hat{\beta}^R := j^R\circ\beta^R$*, where* $j^L : H^L \hookrightarrow \hat{H}^L$ *and* $j^R : H^R \hookrightarrow \hat{H}^R$ *are the canonical inclusions.*

Proof. The equivalence of the two statements in (ii) is evident. By Theorem 4 (ii), the satisfication of the formulas (32) and (33) uniquely determines $\Delta^L(P)$ and $\Delta^R(Q)$, showing the uniqueness. To show the existence, we set the values of Δ^L and Δ^R by (34) and check that (32) and (33) hold. We already know this for $P, Q \in \hat{S}(\mathfrak{g}^*)$ by (27) and (27). Using the action axiom for $\blacktriangleright$, observe that

$$\begin{array}{rcl} \hat{x}_\mu \blacktriangleright (P \blacktriangleright (\hat{f}\hat{g})) & = & \hat{x}_\mu \cdot m(\Delta^L(P)(\blacktriangleright\otimes\blacktriangleright)(\hat{f}\otimes\hat{g})) \\ & = & m(\hat{x}_\mu\Delta^L(P)(\blacktriangleright\otimes\blacktriangleright)(\hat{f}\otimes\hat{g})) \\ & \overset{(34)}{=} & m(\Delta^L(\hat{x}_\mu P)(\blacktriangleright\otimes\blacktriangleright)(\hat{f}\otimes\hat{g})) \end{array}$$

for all $\hat{f}, \hat{g} \in \mathcal{A}^L$, hence (32) holds for all $P \in H^L$. Likewise check (33) for all $Q \in H^R$. Thus, (i). The statement in (ii) that Δ^L, Δ^R then extend $\Delta_{\hat{S}(\mathfrak{g}^*)}, \Delta^{\mathrm{op}}_{\hat{S}(\mathfrak{g}^*)}$ is the statement that (32),(33) specialize to (27),(28) when $P, Q \in \hat{S}(\mathfrak{g}^*)$. The rest of (ii) follows from uniqueness in (i).

(iii) By Theorem 4 (ii), the first 4 formulas follow from (17),(22), (18),(23). The formulas for $\Delta^L(\hat{y}_\alpha)$ and $\Delta^R(\hat{x}_\alpha)$ are straightforward.

(iv) To show that Δ^L is an $\mathcal{A}^L$-bimodule map note that by (ii) Δ^L commutes with the left $\mathcal{A}^L$-action. It remains to show that Δ^L also commutes with the right $\mathcal{A}^L$-action. This is sufficient to check on the generators $\hat{x}_\mu$ of $\mathcal{A}^L$ and arbitrary $P \in \hat{S}(\mathfrak{g}^*)$:

$$\begin{aligned}
\Delta^L(P).\hat{x}_\mu &= \sum P_{(1)} \otimes_{\mathcal{A}^L} \beta(\hat{x}_\mu) P_{(2)} \\
&= \sum P_{(1)} \otimes_{\mathcal{A}^L} \alpha(\hat{x}_\nu)(\mathcal{O}^{-1})^\nu_\mu P_{(2)} \\
&= \sum \beta(\hat{x}_\nu) P_{(1)} \otimes_{\mathcal{A}^L} (\mathcal{O}^{-1})^\nu_\mu P_{(2)} \\
&= \sum \hat{x}_\gamma (\mathcal{O}^{-1})^\gamma_\nu P_{(1)} \otimes_{\mathcal{A}^L} (\mathcal{O}^{-1})^\nu_\mu P_{(2)} \\
&= \Delta^L(\hat{x}_\nu (\mathcal{O}^{-1})^\nu_\mu P) \\
&= \Delta^L(\beta(\hat{x}_\mu)(P))
\end{aligned}$$

By Theorem 4 (iii) for $r = 3$, the action axiom for $\blacktriangleright$ and associativity in H^L implies the coassociativity of Δ^L.

We exhibit the counits ϵ^L and ϵ^R (and their completed versions $\hat{\epsilon}^L : \hat{H}^L \to \mathcal{A}^L$, $\hat{\epsilon}^R : \hat{H}^R \to \mathcal{A}^R$) by the corresponding actions on 1,

$$\epsilon^L(h) := h \blacktriangleright 1_{\mathcal{A}^L}, \quad \epsilon^R(h) := 1_{\mathcal{A}^R} \blacktriangleleft h. \tag{36}$$

The counit axioms (35) for ϵ^L are checked on the generators $\hat{x}_\mu$:

$$\sum \alpha(\epsilon^L(\hat{x}_{\mu(1)}))\hat{x}_{\mu(2)} = \alpha(\epsilon^L(\hat{x}_\mu))1 = \hat{x}_\mu,$$
$$\sum \beta(\epsilon^L(\hat{x}_{\mu(2)}))\hat{x}_{\mu(1)} = \beta(\epsilon^L(1))\hat{x}_\mu = \hat{x}_\mu.$$

Similarly, one checks the counit identities for ϵ^R.

Using formal expressions in the completions, (v) is straightforward.

DEFINITION 5. *(Modification of [2, 4, 6]). Given an algebra $\mathcal{A}$, a* **formally completed left $\mathcal{A}$-bialgebroid** $(H, m, \alpha, \beta, \Delta, \epsilon)$ *consists of the following data. (H, m) which is a complete cofiltered vector space with multiplication distributive with respect to the formal sums in each argument (Appendix A.2); $\alpha : \mathcal{A} \to H$ and $\beta : \mathcal{A}^{\mathrm{op}} \to H$ are fixed algebra homomorphisms with commuting images; H is equipped with a structure of an $\mathcal{A}$-bimodule via the formula $a.h.a' := \alpha(a)\beta(a')h$; $\Delta : \mathcal{A} \to H\hat{\otimes}_{\mathcal{A}}H$ is an $\mathcal{A}$-bimodule map, coassociative and with counit $\epsilon : H \to \mathcal{A}$ understood with respect to the completed tensor product $\hat{\otimes}$ (for the purposes of examples in our mind, we may also require that Δ is a strict morphism of cofiltered algebras, so that (H, Δ, ϵ) is an internal $\mathcal{A}$-coring, with trivial cofiltration understood on $\mathcal{A}$). It is required that*

(i) ϵ is a **left character** *on the $\mathcal{A}$-ring (H, m, α) in the sense that the formula $h \otimes \hat{f} \mapsto \epsilon(h\alpha(\hat{f}))$ defines an action $H \otimes \mathcal{A} \to \mathcal{A}$ extending the left regular action $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$;*

(ii) the coproduct $\Delta : H \to H \otimes_{\mathcal{A}} H$ corestricts to the **completed Takeuchi product**

$$H \hat{\times}_{\mathcal{A}} H \subset H \hat{\otimes}_{\mathcal{A}} H$$

which is by definition the completion of the $\mathcal{A}$-subbimodule,

$$H \times_{\mathcal{A}} H = \{\sum_i b_i \otimes b'_i \in H \otimes_{\mathcal{A}} H \mid \sum_i b_i \otimes b'_i \alpha(a) = \sum_i b_i \beta(a) \otimes b'_i, \ \forall a \in \mathcal{A}\}$$

of $H\hat{\otimes}_{\mathcal{A}}H$ and (by extending the classical reasoning on $H\times_{\mathcal{A}}H$ to the completion, [2, 21]) is an algebra with factorwise multiplication.

(iii) The corestriction $\Delta| : H \to H\hat{\times}_{\mathcal{A}}H$ is an algebra map.

Notice that, because $I \subset H\otimes H$ is just a right ideal in general, the tensor product $H\otimes_{\mathcal{A}}H = H\otimes H/I$ does not carry a well-defined multiplication induced from $H\otimes H$, unlike $H\times_{\mathcal{A}}H$ which does.

Interchanging the left and right sides in all modules and binary tensor products in the definition of a left $\mathcal{A}$-bialgebroid, we get a **right $\mathcal{A}$-bialgebroid** [2]. The $\mathcal{A}$-bimodule structure on H is then given by $a.h.b := h\alpha(b)\beta(a)$. In short, $(H, m, \alpha, \beta, \Delta, \epsilon)$ is a right $\mathcal{A}$-bialgebroid iff $(H, m, \beta, \alpha, \Delta^{\mathrm{op}}, \epsilon)$ is a left $\mathcal{A}^{\mathrm{op}}$-bialgebroid. H^L is not quite an $\mathcal{A}^L$-bialgebroid: Δ^L takes value in the completion $H^L\hat{\otimes}_{\mathcal{A}^L}H^L$. Moreover, to check the compatibility of Δ^L with the multiplication m, we seem to need to extend m to $(H^L\hat{\otimes}H^L)\otimes(H^L\hat{\otimes}H^L)\to H^L\hat{\otimes}H^L$ (to induce the multiplication on completed Takeuchi). Now, recall that $H^L\hat{\otimes}H^L \cong \hat{H}^L\hat{\otimes}\hat{H}^L$. Thus, our setup seem to suggest that, in addition to $\hat{\otimes}$ for Δ^L, we should replace H^L by the completed smash product $(\hat{H}^L, \hat{m})$ from Theorem [**?**] (and Proposition 2) (but a modified framework in [18] gives an alternative).

PROPOSITION 4. *$(\hat{H}^L, \hat{m}, \hat{\alpha}^L, \hat{\beta}^L, \hat{\Delta}^L, \hat{\epsilon}^L)$ has a structure of formally completed left $\mathcal{A}^L$-bialgebroid and $(\hat{H}^R, \hat{m}, \hat{\alpha}^R, \hat{\beta}^R, \hat{\Delta}^R, \hat{\epsilon}^R)$ of a formally completed right $\mathcal{A}^R$-bialgebroid.*

Proof. The internal coring axioms are checked in Proposition 3.

To check that $\sum_\lambda h_\lambda\otimes\hat{f}_\lambda \mapsto \sum_\lambda \epsilon^L(h_\lambda\alpha(\hat{f}_\lambda))$ is an action and (i) holds for ϵ^L, observe from the definition (36) that $\epsilon^L(h\alpha(\hat{f})) = h\alpha(\hat{f}) \blacktriangleright 1 = h\blacktriangleright\hat{f}$, for all $\hat{f}\in\mathcal{A}^L$, $h\in H^L$. Analogously check (i) for $\epsilon^R, \tilde{\epsilon}^L, \tilde{\epsilon}^R$.

To show that $\hat{\Delta}^L$ corestricts to the completed Takeuchi product $\hat{H}^L\hat{\times}_{\mathcal{A}^L}\hat{H}^L$, calculate for $P\in\hat{H}^L$ and $\hat{f},\hat{g},\hat{h}\in\mathcal{A}^L$,

$$\begin{array}{rcl}((P_{(1)}\hat{\beta}^L(\hat{g})\blacktriangleright\hat{f})\cdot(P_{(2)}\blacktriangleright\hat{h}) & = & (P_{(1)}\blacktriangleright(\hat{\beta}^L(\hat{g})\blacktriangleright\hat{f}))\cdot(P_{(2)}\blacktriangleright\hat{h}) \\ & \stackrel{(30)}{=} & (P_{(1)}\blacktriangleright(\hat{f}\hat{g}))\cdot(P_{(2)}\blacktriangleright\hat{h}) \\ & = & P\blacktriangleright(\hat{f}\hat{g}\hat{h}) \\ & = & (P_{(1)}\blacktriangleright\hat{f})\cdot((P_{(2)}\hat{\alpha}^L(\hat{g}))\blacktriangleright\hat{h}),\end{array}$$

thus, by Theorem 4 (ii), (iv), $P_{(1)}\hat{\beta}^L(\hat{g})\otimes_{\mathcal{A}^L}P_{(2)} = P_{(1)}\otimes_{\mathcal{A}^L}P_{(2)}\hat{\alpha}^L(\hat{g})$, hence $\hat{\Delta}^L(P)\in\hat{H}^L\hat{\times}_{\mathcal{A}^L}\hat{H}^L$.

We now check directly that the corestriction $\hat{\Delta}^L : \hat{H}^L \to \hat{H}^L\hat{\times}_{\mathcal{A}^L}\hat{H}^L$ is a homomorphism of algebras,

$$\hat{\Delta}^L(h_1h_2) = \hat{\Delta}^L(h_1)\hat{\Delta}^L(h_2) \quad \text{for all } h_1, h_2\in H^L.$$

To this aim, recall that $\Delta_{\hat{S}(\mathfrak{g}^*)} : \hat{S}(\mathfrak{g}^*) \to \hat{S}(\mathfrak{g}^*)\hat{\otimes}\hat{S}(\mathfrak{g}^*)$ is a homomorphism, and that by Proposition 3 (ii), $\hat{\Delta}^L|_{1\sharp\hat{S}(\mathfrak{g}^*)}$ is the composition

$$1\sharp\hat{S}(\mathfrak{g}^*) \cong \hat{S}(\mathfrak{g}^*) \stackrel{\Delta_{\hat{S}(\mathfrak{g}^*)}}{\longrightarrow} \hat{S}(\mathfrak{g}^*)\hat{\otimes}\hat{S}(\mathfrak{g}^*) \hookrightarrow \hat{H}^L \hat{\times}_{\mathcal{A}^L} \hat{H}^L,$$

hence homomorphism as well (the inclusion is a homomorphism, because the product is factorwise). We use this when applying to the tensor factor $P_{(2)}Q$ in the calculation

$$\begin{array}{rcl}
\hat{\Delta}^L((u\sharp P)(v\sharp Q)) & = & \hat{\Delta}^L(u(P_{(1)} \blacktriangleright v)\sharp P_{(2)}Q) \\
& = & (u(P_{(1)} \blacktriangleright v)\sharp P_{(2)}Q_{(1)}) \otimes (1\sharp P_{(3)}Q_{(2)}). \\
& = & [(u\sharp P_{(1)})(v\sharp Q_{(1)})] \otimes (1\sharp P_{(2)}Q_{(2)}) \\
& = & (u(P_{(1)} \blacktriangleright v)\sharp P_{(2)}Q_{(1)}) \otimes (1\sharp P_{(3)}Q_{(2)}). \\
& = & [(u\sharp P_{(1)}) \otimes (1\sharp P_{(2)})][(v\sharp Q_{(1)}) \otimes (1\sharp Q_{(2)})] \\
& = & \hat{\Delta}^L(u\sharp P)\hat{\Delta}^L(v\sharp Q).
\end{array}$$

6. The antipode and Hopf algebroid

A Hopf algebroid is roughly a bialgebroid with an antipode. In the literature, there are several nonequivalent versions. In the framework of G. Böhm [2], there are two variants which are equivalent if the antipode is bijective (as it is here the case): nonsymmetric and symmetric. The *nonsymmetric* involves one-sided bialgebroid with an antipode map satisfying axioms which involve both the antipode map and its inverse. The *symmetric* version involves two bialgebroids and axioms neither involve nor require the inverse of the antipode. We choose this version here, because we naturally constructed two actions, $\blacktriangleright$ and $\blacktriangleleft$, which lead to the two coproducts, Δ^L and Δ^R, as exhibited in Section 5.

DEFINITION 6. *Given two algebras $\mathcal{A}^L$ and $\mathcal{A}^R$ with fixed isomorphism $(\mathcal{A}^L)^{\mathrm{op}} \cong \mathcal{A}^R$, a* **symmetric Hopf algebroid** *([2]) is a pair of a left $\mathcal{A}^L$-bialgebroid H^L and a right $\mathcal{A}^R$-bialgebroid H^R, isomorphic and identified as algebras $H \cong H^L \cong H^R$, such that the compatibilities*

$$\begin{array}{c}
\alpha^L \circ \epsilon^L \circ \beta^R = \beta^R, \;\; \beta^L \circ \epsilon^L \circ \alpha^R = \alpha^R, \\
\alpha^R \circ \epsilon^R \circ \beta^L = \beta^L, \;\; \beta^R \circ \epsilon^R \circ \alpha^L = \alpha^L,
\end{array} \tag{37}$$

hold between the source and target maps $\alpha^L, \alpha^R, \beta^L, \beta^R$, and the counits ϵ^L, ϵ^R; the comultiplications Δ^L and Δ^R satisfy the compatibility relations

$$(\Delta^R \otimes_{\mathcal{A}^L} \mathrm{id}) \circ \Delta^L = (\mathrm{id} \otimes_{\mathcal{A}^R} \Delta^L) \circ \Delta^R \tag{38}$$

$$(\Delta^L \otimes_{\mathcal{A}^R} \mathrm{id}) \circ \Delta^R = (\mathrm{id} \otimes_{\mathcal{A}^L} \Delta^R) \circ \Delta^L \tag{39}$$

and there is a map $\mathcal{S} : H \to H$, called the **antipode** *which is an antihomomorphism of algebras and satisfies*

$$\begin{aligned} \mathcal{S} \circ \beta^L = \alpha^L, \qquad \mathcal{S} \circ \beta^R = \alpha^R \\ m \circ (\mathcal{S} \otimes id) \circ \Delta^L = \alpha^R \circ \epsilon^R \\ m \circ (id \otimes \mathcal{S}) \circ \Delta^R = \alpha^L \circ \epsilon^L \end{aligned} \tag{40}$$

A formally completed symmetric Hopf algebroid is defined analogously replacing the tensor product with the completed tensor product in the axioms above.

THEOREM 5. *Data $\mathcal{A}^L = U(\mathfrak{g}^L)$, $\mathcal{A}^R = U(\mathfrak{g}^R)$, $\hat{H}^L := U(\mathfrak{g}^L)\hat{\sharp}\hat{S}(\mathfrak{g}^*)$, $\hat{H}^R := \hat{S}(\mathfrak{g}^*)\hat{\sharp}U(\mathfrak{g}^R)$, $\hat{\epsilon}^L, \hat{\epsilon}^R, \hat{\alpha}^L, \hat{\beta}^L, \hat{\alpha}^R, \hat{\beta}^R$ from Section 4 and $\hat{\Delta}^L$, $\hat{\Delta}^R$ defined in Section 5, satisfy the axioms for Hopf algebroid provided the tensor product in the axioms is replaced by the completed tensor product. The antipode map $\mathcal{S} : \hat{H} \to \hat{H}$ is a unique antihomomorphism of algebras distributive over formal sums such that*

$$\mathcal{S}(\partial^\nu) = -\partial^\nu,$$

(hence, by continuity, $\mathcal{S}(\mathcal{O}) = \mathcal{S}(e^{\mathcal{C}}) = e^{-\mathcal{C}} = \mathcal{O}^{-1}$), and

$$\mathcal{S}(\hat{y}_\mu) = \hat{x}_\mu. \tag{41}$$

The antipode $\mathcal{S}$ is bijective. For general $\mathfrak{g}$, $\mathcal{S}^2 \neq \mathrm{id}$. More precisely,

$$\mathcal{S}^2(\hat{y}_\mu) = \mathcal{S}(\hat{x}_\mu) = \hat{y}_\mu - C^\lambda_{\mu\lambda}, \quad \mathcal{S}^{-2}(\hat{x}_\mu) = \mathcal{S}^{-1}(\hat{y}_\mu) = \hat{x}_\mu - C^\lambda_{\mu\lambda} \tag{42}$$

$$\mathcal{S}^2(\hat{x}_\mu) = \hat{x}_\mu + C^\lambda_{\mu\lambda}, \quad \mathcal{S}^{-2}(\hat{y}_\mu) = \hat{y}_\mu + C^\lambda_{\mu\lambda}. \tag{43}$$

Proof. In this proof, we simply write ϵ^L, Δ^L etc. without hat symbol, as it is not essential for the arguments below.

One checks the relations (37) on generators, for which $\alpha^R(\hat{y}_\mu) = \hat{y}_\mu$, $\beta^R(\hat{y}_\mu) = \mathcal{O}^\rho_\mu \hat{y}_\rho = \mathcal{O}^\rho_\mu \hat{y}_\sigma (\mathcal{O}^{-1})^\sigma_\rho$, $\alpha^L(\hat{x}_\mu) = \hat{x}_\mu$, $\beta^L(\hat{x}_\mu) = \hat{y}_\mu$.

Regarding that Δ^L and Δ^R restricted to $\hat{S}(\mathfrak{g}^*)$ coincide with $\Delta_{\hat{S}(\mathfrak{g}^*)}$, (38) and (39) restricted to $\hat{S}(\mathfrak{g}^*)$ reduce to the coassociativity. Algebra H^L is generated by $\hat{S}(\mathfrak{g}^*)$ and $\mathfrak{g}^L$, so it is enough to check (38),(39) also on $\hat{y}_\mu = \hat{x}_\nu (\mathcal{O}^{-1})^\nu_\mu$. This follows from the matrix identities

$$\Delta^L(\hat{x}\mathcal{O}^{-1}) = \hat{x}\mathcal{O}^{-1} \otimes_{\mathcal{A}^L} \mathcal{O}^{-1} = \mathcal{O}^{-1}\mathcal{O} \otimes_{\mathcal{A}^L} \hat{x}\mathcal{O}^{-1} = 1 \otimes_{\mathcal{A}^L} \hat{y},$$

$$\Delta^R(\hat{y}) = 1 \otimes_{\mathcal{A}^R} \hat{y} = \hat{y} \otimes_{\mathcal{A}^R} \mathcal{O}^{-1} = \hat{x}\mathcal{O}^{-1} \otimes_{\mathcal{A}^R} \mathcal{O}^{-1}.$$

Formula $\mathcal{S}(\partial^\mu) = -\partial^\mu$ clearly extends to a unique continuous antihomomorphism of algebras on the formal power series ring $\hat{S}(\mathfrak{g}^*)$.

Similarly, by functoriality of $\mathfrak{g} \mapsto U(\mathfrak{g})$, the antihomomorphism of Lie algebras, $\mathcal{S} : \mathfrak{g}^R \to \mathfrak{g}^L$, $\hat{y}_\mu \mapsto \hat{x}_\mu$, extends to a unique antihomomorphism $U(\mathfrak{g}^R) \to U(\mathfrak{g}^L)$. Regarding that $U(\mathfrak{g}^R)$ and $\hat{S}(\mathfrak{g}^*)$ generate H^R, it is sufficient to check that $\mathcal{S}$ is compatible with the additional relations in the smash product, namely $[\partial^\mu, \hat{y}_\nu] = \left(\frac{\mathcal{C}}{e^{\mathcal{C}}-1}\right)^\mu_\nu$. Then $\mathcal{S}([\partial^\mu, \hat{x}_\nu]) = \mathcal{S}\left(\frac{-\mathcal{C}}{e^{-\mathcal{C}}-1}\right)^\mu_\nu = \left(\frac{\mathcal{C}}{e^{\mathcal{C}}-1}\right)^\mu_\nu = \left(e^{-\mathcal{C}}\frac{-\mathcal{C}}{e^{-\mathcal{C}}-1}\right)^\mu_\nu$, which equals $(e^{-\mathcal{C}})^\rho_\nu[\hat{x}_\rho, -\partial^\mu] = [\mathcal{S}(\hat{y}_\rho\mathcal{O}^\rho_\nu), -\partial^\mu] = [\mathcal{S}(\hat{x}_\nu), \mathcal{S}(\partial^\mu)]$.

To exhibit the inverse $\mathcal{S}^{-1}$, we similarly check that the obvious formulas $\mathcal{S}^{-1}(\hat{x}_\mu) = \hat{y}_\mu$, $\mathcal{S}^{-1}(\partial^\mu) = \partial^\mu$ define a unique continuous antihomomorphism $\mathcal{S}^{-1} : H \to H$.

For (42) calculate $\mathcal{S}(\hat{x}_\mu) = \mathcal{S}(\hat{y}_\rho\mathcal{O}^\rho_\mu) = \mathcal{S}(\mathcal{O}^\rho_\mu)\mathcal{S}(\hat{y}_\rho) = (\mathcal{O}^{-1})^\rho_\mu\hat{x}_\rho = (\mathcal{O}^{-1})^\rho_\mu\hat{y}_\sigma\mathcal{O}^\sigma_\rho$ and use $[\mathcal{O}^\rho_\mu, \hat{y}_\sigma] = -C^\rho_{\tau\sigma}\mathcal{O}^\tau_\mu$ in the last step. Similarly, we get $\mathcal{S}^{-1}(\hat{y}_\mu) = \mathcal{O}^\rho_\mu\hat{x}_\sigma(\mathcal{O}^{-1})^\sigma_\rho$ and use $[\mathcal{O}^\rho_\mu, \hat{x}_\sigma] = -C^\rho_{\tau\sigma}\mathcal{O}^\tau_\mu$ for the second formula in (42). Notice that $\mathcal{S}^{-1}(\hat{y}_\mu) = \hat{z}_\mu$ from Theorem 3, formula (24). For (43) similarly use the matrix identities $\mathcal{S}^2(\hat{x}) = \mathcal{S}(\mathcal{O}^{-1}\hat{y}\mathcal{O}) = \mathcal{O}^{-1}\hat{x}\mathcal{O}$, $\mathcal{S}^{-2}(\hat{y}) = \mathcal{O}\hat{y}\mathcal{O}^{-1}$.

The formula $\mathcal{S}(\beta^L(\hat{x}_\mu)) = \mathcal{S}(\hat{y}_\mu) = \hat{x}_\mu = \alpha^L(\hat{x}_\mu)$ shows $\mathcal{S} \circ \beta^L = \alpha^L$ for the generators of $\mathcal{A}^L$. Likewise for the rest of the identities (40).

7. Conclusion and perspectives.

We have equipped the noncommutative phase spaces of Lie algebra type with the structure of a version of a Hopf algebroid over $U(\mathfrak{g})$. That means that we have found a left $U(\mathfrak{g})$-bialgebroid $\hat{H}^L$, and a right $U(\mathfrak{g})^{\mathrm{op}}$-bialgebroid $\hat{H}^R$, which are canonically isomorphic as associative algebras $\hat{H}^L \cong \hat{H}^R$, and an antipode map $\mathcal{S}$ satisfying a number of axioms involving a completed tensor product $\hat{\otimes}$.

Hopf algebroids allow a version of Drinfeld's twisting cocycles studied earlier in the context of deformation quantization [22], and are a promising tool for extending many constructions to the noncommutative case, and a planned direction for our future work. One can find a cocycle which can be used to twist the Hopf algebroid corresponding to the abelian Lie algebra (i.e. the Hopf algebroid structure on the completion of the usual Weyl algebra) to recover the Hopf algebroid of the phase space for any other Lie algebra of the same dimension. More importantly for applications, along with the phase space one can systematically twist many geometric structures, including differential forms, from the undeformed to the deformed case. This has earlier been studied in the case of κ-spaces (e.g. in [12]), while the work for general finite-dimensional Lie algebras (and for some nonlinear star products) is in progress.

Acknowledgments. We thank L. El Kaoutit, V. Roubtsov, T. Brzeziński, T. Maszczyk and G. Böhm for reading fragments of this work and advice. We thank A. Borowiec, J. Lukierski and A. Pachoł for discussions. Most of the work has been done at IRB, Zagreb, but substantial progress has also been made during the visit of Z.Š. to IHÉS in November 2012. Z.Š. thanks G. Garkusha for the invitation to present a related *Algebra and topology seminar* at Swansea, July 15, 2014.

Appendix

A.1 Commutation $[\hat{x}_\alpha, \hat{y}_\beta] = 0$

PROPOSITION 5. *The identity $[\hat{x}_\mu, \hat{y}_\nu] = 0$ holds in the realization $\hat{x}_\mu = x_\sigma \phi^\sigma_\mu = x_\sigma \left(\frac{-\mathcal{C}}{e^{-\mathcal{C}}-1}\right)^\sigma_\mu$, $\hat{y}_\mu = x_\rho \tilde{\phi}^\rho_\mu = x_\rho \left(\frac{\mathcal{C}}{e^{\mathcal{C}}-1}\right)^\rho_\mu$, where $\mathcal{C}^\mu_\nu = C^\mu_{\nu\gamma}\partial^\gamma$ (cf. the equations (8,5,6)).*

Proof. For any formal series $P = P(\partial)$ in ∂-s, $[P, \hat{x}_\mu] = \frac{\partial P}{\partial(\partial^\mu)} =: \delta_\mu P$. In particular (cf. [8]), from $[\hat{x}_\mu, \hat{x}_\nu] = C^\lambda_{\mu\nu}\hat{x}_\lambda$, one obtains a formal differential equation for ϕ^σ_μ,

$$(\delta_\rho \phi^\gamma_\mu)\phi^\rho_\nu - (\delta_\rho \phi^\gamma_\nu)\phi^\rho_\mu = C^\sigma_{\mu\nu}\phi^\gamma_\sigma. \tag{44}$$

By symmetry $C^i_{jk} \mapsto -C^i_{jk}$ the same equation holds with $(-\tilde{\phi}) = \frac{-\mathcal{C}}{e^{\mathcal{C}}-1}$ in the place of ϕ. Similarly, the equation $[\hat{x}_\mu, \hat{y}_\nu] = 0$, i.e. $[x_\gamma \phi^\gamma_\mu, x_\beta \tilde{\phi}^\beta_\nu] = 0$, is equivalent to

$$(\delta_\rho \phi^\gamma_\mu)\tilde{\phi}^\rho_\nu - (\delta_\rho \tilde{\phi}^\gamma_\nu)\phi^\rho_\mu = 0 \tag{45}$$

Recall that $\phi = \frac{-\mathcal{C}}{e^{-\mathcal{C}}-1} = \sum_{N=0}^\infty (-1)^N \frac{B_N}{N!}(\mathcal{C}^N)^i_j$, where B_N are the Bernoulli numbers, which are zero unless N is either even or $N = 1$. Hence $\tilde{\phi} = \frac{\mathcal{C}}{e^{\mathcal{C}}-1} = \sum_{N=0}^\infty \frac{B_N}{N!}\mathcal{C}^N = \frac{B_1}{2}\mathcal{C} + \sum_{N\,\mathrm{even}}^\infty \frac{B_N}{N!}\mathcal{C}^N$ and $\phi - \tilde{\phi} = -2\frac{B_1}{2}\mathcal{C} = \mathcal{C}$. Notice that $\frac{\partial \mathcal{C}^\alpha_\beta}{\partial(\partial^\mu)} = C^\alpha_{\beta\mu}$. Therefore, subtracting (45) from (44) gives the condition

$$(\delta_\rho \phi^\gamma_\mu)\mathcal{C}^\rho_\nu - C^\gamma_{\nu\rho}\phi^\rho_\mu = C^\sigma_{\mu\nu}\phi^\gamma_\sigma.$$

$\mathcal{C}$ is homogeneous of degree 1 in ∂^μ-s, so we can split this condition into the parts of homogeneity degree N:

$$[\delta_\rho(\mathcal{C}^N)^\gamma_\mu]\mathcal{C}^\rho_\nu - (\delta_\rho \mathcal{C}^\gamma_\nu)(\mathcal{C}^N)^\rho_\mu = C^\sigma_{\mu\nu}(\mathcal{C}^N)^\gamma_\sigma, \tag{46}$$

where the overall factor of $(-1)^N B_N/N!$ has been taken out. Hence the proof is reduced to the following lemma:

LEMMA 2. *The identities (46) hold for $N = 0, 1, 2, \ldots$.*

Proof. For $N = 0$, (46) reads $C^\gamma_{\nu\mu} = C^\gamma_{\mu\nu}$, which is the antisymmetry of the bracket. For $N = 1$ it follows from the Jacobi identity:

$$(C^\gamma_{\mu\rho}C^\rho_{\nu\tau} - C^\gamma_{\nu\rho}C^\rho_{\mu\tau})\partial^\tau = C^\rho_{\mu\nu}C^\gamma_{\rho\tau}\partial^\tau.$$

Suppose now (46) holds for given $N = K \geq 1$. Then

$$C^\gamma_{\mu\nu}(\mathcal{C}^K)^\rho_\sigma\mathcal{C}^\gamma_\rho = [\delta_\rho(\mathcal{C}^K)^\rho_\mu]\mathcal{C}^\sigma_\nu\mathcal{C}^\gamma_\rho - C^\rho_{\nu\sigma}(\mathcal{C}^K)^\sigma_\mu\mathcal{C}^\gamma_\rho$$

By the usual Leibniz rule for δ_ρ, this yields

$$C^\gamma_{\mu\nu}(\mathcal{C}^K)^\rho_\sigma\mathcal{C}^\gamma_\rho = \delta_\rho(\mathcal{C}^{K+1})^\gamma_\rho\mathcal{C}^\sigma_\nu - (\mathcal{C}^K)^\rho_\mu C^\gamma_{\rho\sigma}\mathcal{C}^\sigma_\nu - C^\rho_{\nu\sigma}(\mathcal{C}^K)^\sigma_\mu\mathcal{C}^\gamma_\rho.$$

The identity (46) follows for $N = K + 1$ if the second and third summand on the right hand side add up to $-C^\gamma_{\nu\sigma}(\mathcal{C}^{K+1})^\sigma_\mu$. After renaming the indices, one brings the sum of these two to the form

$$(\mathcal{C}^K)^\rho_\mu(-C^\sigma_{\nu\lambda}C^\gamma_{\rho\sigma} + C^\sigma_{\nu\rho}C^\gamma_{\lambda\sigma})\partial^\lambda = -(C^K)^\rho_\mu C^\sigma_{\rho\lambda}\partial^\lambda C^\gamma_{\nu\sigma} = -(C^{K+1})^\sigma_\mu C^\gamma_{\nu\sigma}$$

as required. The Jacobi identity is used for the equality on the left.

A.2 Cofiltered vector spaces and completions

We sketch the formalism treating the algebraic duals $U(\mathfrak{g})^*$ and $S(\mathfrak{g})^*$ of filtered algebras $U(\mathfrak{g}), S(\mathfrak{g})$ as cofiltered algebras. The reader can alternatively treat them as topological algebras: the basis of neighborhoods of 0 in the formal adic topology of $U(\mathfrak{g})^*$ and $S(\mathfrak{g})^*$ is given by the annihilator ideals Ann $U_i(\mathfrak{g})$ and Ann $S_i(\mathfrak{g})$, consisting of functionals vanishing on the i-th filtered component. A **cofiltration** on a vector space A is the (inverse) sequence of epimorphisms of its quotient spaces $\ldots \to A_{i+1} \to A_i \to A_{i-1} \to \ldots \to A_0$; denoting the quotient maps $\pi_i : A \to A_i$ and $\pi_{i,i+k} : A_{i+k} \to A_i$, the identities $\pi_i = \pi_{i,i+k} \circ \pi_{i+k}$, $\pi_{i,i+k+l} = \pi_{i.i+k} \circ \pi_{i+k,i+k+l}$ are required to hold. The limit $\varprojlim_r A_r$ consists of the sequences $(a_r)_{r\in\mathbb{N}_0} \in \prod_r A_r$ of compatible elements, $a_r = \pi_{r,r+k}(a_{r+k})$. The canonical map $A \to \hat{A}$ to the **completion** $\hat{A} := \varprojlim_i A_i$ is 1-1 if $\forall a \in A\ \exists r \in \mathbb{N}_0$ such that $\pi_r(a) \neq 0$. The cofiltration is **complete** if this map is an isomorphism. **Strict morphisms** of cofiltered vector spaces are the linear maps which induce the maps on the quotients. (This makes the category of complete cofiltered vector spaces more rigid than the category of pro-vector spaces.) We say that $a = (a_r)_r$ has the **cofiltered degree** $\geq N$ if $a_r = 0$ for $r < N$. In our main example, $U_i(\mathfrak{g})^* := (U(\mathfrak{g})^*)_i := U(\mathfrak{g})^*/\,\mathrm{Ann}\ U_i(\mathfrak{g}) \cong (U_i(\mathfrak{g}))^*$ and similarly for $S(\mathfrak{g})^* \cong \hat{S}(\mathfrak{g}^*)$. We use lower indices both for filtrations and for cofiltrations (but upper for gradations!).

The usual tensor product $A \otimes B$ of cofiltered vector spaces is cofiltered with r-th cofiltered component (see [18])

$$(A \otimes B)_r = \frac{A \otimes B}{\cap_{p+q=r} \ker \pi^A_p \otimes \ker \pi^B_q}. \tag{47}$$

$(A\otimes B)_r$ is an abelian group of finite sums of the form $\sum_\lambda a_\lambda \otimes b_\lambda \in A \otimes B$ modulo the additive relation of equivalence $\sim_r$ for which $\sum a_\mu \otimes b_\mu \sim_r 0$ iff $\pi_p(a_\mu)\otimes\pi_q(b_\mu) = 0$ in $A_p \otimes B_q$ for all p, q such that $p+q = r$. Define the **completed tensor product** $A\hat{\otimes}B = \varprojlim_r (A\otimes B)_r$, equipped with the same cofiltration, $(A\hat{\otimes}B)_r := (A\otimes B)_r$. The element in $A\hat{\otimes}B$ is thus the class of equivalence of a formal sum $\sum_\lambda a_\lambda \otimes b_\lambda$ such that for any p and q there are at most finitely many λ such that $\pi^A_p(a_\lambda) \otimes \pi^B_q(b_\lambda) \neq 0$. Alternatively, we can equip $A \otimes B$ with bicofiltration ($\mathbb{N}_0 \times \mathbb{N}_0$-cofiltration), $(A \otimes B)_{r,s} = A_r \otimes B_s$. Regarding that for all r, s, there are inclusions $\ker \pi_{r+s} \otimes \ker \pi_{r+s} \subset \cap_{p+q=r+s} \ker \pi_p \otimes \ker \pi_q \subset \ker \pi_r \otimes \ker \pi_s$, there are projections $A_{r+s} \otimes B_{r+s} \twoheadrightarrow (A \otimes B)_{r+s} \twoheadrightarrow A_r \otimes B_s$; therefore the completion with respect to the bicofiltration and with respect to the cofiltration are equivalent (and alike for the convergence of infinite sums inside $A\hat{\otimes}B$). A linear map among cofiltered vector spaces is **distributive over formal sums** if it sends summand by summand formally converging infinite sums to formally converging sums (formal version of σ-additivity). This property is weaker than being a strict morphism of complete cofiltered vector spaces. If A and B are complete, we can consider maps $A \otimes B \to C$ distributive over formal sums in each argument. Unlike morphisms of cofiltered spaces (even in a weak sense [18]), such a map does not need to extend to a map $A\hat{\otimes}B \to \hat{C}$ distributive over formal sums in $A\hat{\otimes}B$ (continuity in each argument separately does not imply joint continuity).

A **cofiltered algebra** A (e.g. $\hat{S}(\mathfrak{g}^*)$) is a monoid internal to the $\boldsymbol{k}$-linear category of complete cofiltered vector spaces, strict morphisms and with the tensor product $\hat{\otimes}$ [18]. The bilinear associative unital multiplication map $\hat{m} : A\hat{\otimes}A \to A$ is a strict morphism, hence inducing linear maps $m_r : (A \otimes A)_r \to A_r$ for all r. In other words, $A\hat{\otimes}A \ni \sum_\lambda a_\lambda \otimes b_\lambda \overset{\hat{m}}{\mapsto} \sum_\lambda a_\lambda \cdot b_\lambda \in A$, where $(\sum_\lambda a_\lambda \cdot b_\lambda)_r$ is an equivalence class in A_r of $(\pi_r \circ \hat{m})(\sum'_\lambda a_\lambda \otimes b_\lambda)$, where $\sum'$ denotes the *finite sum* over all λ such that $\exists(p, q)$ with $p + q = r$ and $\pi_p(a_\lambda) \otimes \pi_q(b_\lambda) \neq 0$.

A vector subspace W of a cofiltered vector space V is cofiltered by $W_p := V_p \cap W$ with a canonical linear map $\varprojlim W_p \to \varprojlim V_p = \hat{V}$, whose image is a cofiltered subspace $\hat{W}_{\hat{V}} \subset \hat{V}$, the **completion** of W

in $\hat{V}$. This is compatible with many additional structures, so defining the completions of sub(bi)modules and ideals (thus $\hat{I}$, $\hat{I}'$, $\hat{I}^{(r)}$, $\hat{I}'^{(r)}$, $\hat{\hat{I}}$, $\hat{\hat{I}}'$, $\hat{\hat{I}}^{(r)}$, $\hat{\hat{I}}'^{(r)}$ in Sections 4 and 5). If U is an ordinary algebra, A_U a right U-module and ${}_UB$ a left U-module, which are cofiltered, then define $A\hat{\otimes}_U B$ as the quotient of $A\hat{\otimes}B$ by the completion of $\ker(A\otimes B\to A\otimes_U B)$ in $A\hat{\otimes}B$.

In this article, the completed tensor product $U(\mathfrak{g}^L)\hat{\otimes}\hat{S}(\mathfrak{g}^*)$, is interpreted by equipping the filtered ring $U(\mathfrak{g}^L)$ with the *trivial cofiltration* $U(\mathfrak{g}^L)$, in which every cofiltered component is the entire $U(\mathfrak{g})$ (and carries the discrete topology). The elements of $U(\mathfrak{g}^L)\hat{\otimes}\hat{S}(\mathfrak{g}^*)$ are given by the formal sums $\sum u_\lambda\otimes a_\lambda$ such that $\forall r$, $\pi_r(a_\lambda)=0$ for all but finitely many λ. The basis of neighborhoods of 0 in $U(\mathfrak{g}^L)\hat{\otimes}\hat{S}(\mathfrak{g}^*)$ consists of the subspaces $\boldsymbol{k}\hat{f}\otimes\prod_{p>r}S^p(\mathfrak{g}^*)$ for all $\hat{f}\in U(\mathfrak{g})$ and $r\in\mathbb{N}$. The right Hopf action $a\otimes\hat{u}\mapsto\boldsymbol{\phi}(\hat{u})(a)$ *admits* a completed smash product:

THEOREM 6. *The multiplication in $H^L=U(\mathfrak{g}^L)\sharp\hat{S}(\mathfrak{g}^*)$ extends to a unique multiplication $\hat{m}$ on $U(\mathfrak{g}^L)\hat{\otimes}\hat{S}(\mathfrak{g}^*)$ which distributes with respect to formal sums in each argument, the* **completed smash product algebra** $\hat{H}^L=U(\mathfrak{g}^L)\hat{\sharp}\hat{S}(\mathfrak{g}^*)$. *It induces a componentwise algebra structure on $\hat{H}^L\hat{\otimes}\hat{H}^L$ distributing over formal sums. However, there are no cofiltered algebra structures on any of the two.*

Proof. The extended multiplication is well defined by a formal sum $\sum_{\lambda,\mu}(u_\lambda\sharp a_\lambda)(u'_\mu\sharp a'_\mu)=\sum_{\lambda,\mu}u_\lambda u'_{\mu(1)}\sharp\boldsymbol{\phi}(u'_{\mu(2)})(a_\lambda)a'_\mu$ if for all $r\in\mathbb{N}_0$ the number of pairs (μ,λ) such that $u_\lambda u'_{\mu(1)}\otimes\pi_r(\boldsymbol{\phi}(u'_{\mu(2)})(a_\lambda)\cdot a'_\mu)\neq 0$ (only Sweedler summation) is finite. There are only finitely many μ such that $\pi_r(a'_\mu)\neq 0$; only those contribute to the sum because $\pi_k(a)\pi_l(b)=0$ implies $\pi_{k+l}(ab)=0$ in any cofiltered ring. For each such μ fix a representation of $\Delta(u_\mu)$ as a finite sum $\sum_k u_{\mu(1)k}\otimes u_{\mu(2)k}$ and denote by $K(\mu)$ the maximal over k filtered degree of $u_{\mu(2)k}$ and by $L(\mu)$ the minimal cofiltered degree of a'_μ. By Lemma 1 (iii) and induction we see that if $a_\lambda\in\hat{S}(\mathfrak{g}^*)_s$ then $\boldsymbol{\phi}(u_{\mu(2)k})(a_\lambda)\in\hat{S}(\mathfrak{g}^*)_{s-K(\mu)}$. Hence for each μ there are only finitely many λ for which $s-K(\mu)+L(\mu)\leq r$. That is sufficient for the conclusion. More details will be exhibited in [18].

References

1. AMELINO-CAMELIA G., ARZANO M.: *Coproduct and star product in field theories on Lie-algebra non-commutative space-times*, Phys. Rev. D65:084044 (2002) `hep-th/0105120`.
2. BÖHM G.: *Hopf algebroids*, in Handbook of Algebra, 44 pages (2008) `arXiv:0805.3806`.

3. Böhm G.: *Internal bialgebroids, entwining structures and corings*, AMS Contemp. Math. 376 (2005) 207-226, `math.QA/0311244`.
4. Böhm G, Szlachányi K.: *Hopf algebroids with bijective antipodes: axioms, integrals and duals*, Comm. Alg. 32 (11) (2004) 4433–4464 (math.QA/0305136)
5. Bourbaki N.: *Lie groups and algebras*, Ch. I-III, Hermann, Paris, 1971 (Ch.I), 1972 (Ch.II-III) (French); Springer 1975, 1989 (Ch. I-III, English).
6. Brzeziński T., Militaru G.: *Bialgebroids, $\times_A$-bialgebras and duality*, J. Alg. 251: 279–294 (2002) `math.QA/0012164`.
7. Brzeziński T., Wisbauer R., Corings and comodules. *London Math. Soc. Lect. Note Ser.,* 309. Cambridge Univ. Press 2003.
8. Durov N., Meljanac S., Samsarov A., Škoda Z.: *A universal formula for representing Lie algebra generators as formal power series with coefficients in the Weyl algebra*, J. Alg. 309, n. 1, 318–359 (2007) `math.RT/0604096`.
9. T. R. Govindarajan, K. S. Gupta, E. Harikumar, S. Meljanac, D. Meljanac, Twisted statistics in κ-Minkowski spacetime, Phys. Rev. D 77 (2008), no. 10, 105010, 6 pp.
10. Halliday S., Szabo R. J.: *Noncommutative field theory on homogeneous gravitational waves*, J. Phys. A39 (2006) 5189–5226, `arXiv:hep-th/0602036`.
11. Kontsevich M.: *Deformation quantization of Poisson manifolds*, Lett. Math. Phys. 66 (2003), no. 3, 157–216; `q-alg/9709040`.
12. Jurić T., Kovačević D., Meljanac S., *κ-deformed phase space, Hopf algebroid and twisting*, SIGMA 10 (2014) 106, 1–18; `arXiv:1402.0397`.
13. Jurić T., Meljanac S., Štrajn R.: *κ-Poincaré-Hopf algebra and Hopf algebroid structure of phase space from twist,*, Phys. Lett. A 377 (2013), pp. 2472-2476, `arXiv:1303.0994`; *Twists, realizations and Hopf algebroid structure of kappa-deformed phase space*, Int. J. Mod. Phys. A 29 (2014) 1450022, `arxiv:1305.3088`.
14. Lu J-H.: *Hopf algebroids and quantum groupoids*, Int. J. Math. 7 (1996) 47–70, `q-alg/9505024`.
15. Lukierski J., Nowicki A., Ruegg H., Tolstoy V. N.: *q-deformation of Poincaré algebra*, Phys. Lett. B 264 (1991) 331.
16. Lukierski J., Nowicki A.: *Heisenberg double description of κ-Poincaré algebra and κ-deformed phase space*, in Proc. XXI Int. Coll. Group Theor. Methods in Physics, ed. V.K. Dobrev, H.D. Doebner, Sofia (1997) 186–192, `arXiv:q-alg/9792003`.
17. Meljanac S., Samsarov A., Stojić Marko, Gupta K.S.: *Kappa-Minkowski space-time and the star product realizations*, Eur. Phys. J. C53:295-309 (2008) `arXiv:0705.2471`.
18. Škoda Z., Stojić Martina: *Doubles of filtered Hopf algebras, cofiltrations, and Hopf algebroid of formal differential operators*, in preparation.
19. Meljanac S., Škoda Z.: *Leibniz rules for enveloping algebras and a diagrammatic expansion*, `arXiv:0711.0149`; with newer version at `http://www.irb.hr/korisnici/zskoda/scopr8.pdf`.
20. Škoda Z.: *Heisenberg double versus deformed derivatives*, Int. J. Mod. Phys. A 26, Nos. 27 & 28 (2011) 4845–4854, `arXiv:0806.0978`.
21. Takeuchi, M.: *Groups of algebras over $A \otimes \bar{A}$*, J. Math. Soc. Japan 29:3 (1977), 459–492.
22. Xu, P.: *Quantum groupoids*, Commun. Math. Phys., 216:539–581 (2001) `q-alg/9905192`.