Dyson-Schwinger equations for the auxiliary field formulation of the $O(N)$ model

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Plan of the talk:

- The auxiliary field formulation to $\mathcal{O}(1/N)$ accuracy:
  Hubbard-Stratonovich transformation,
  Dyson-Schwinger equations
- The leading order (LO) solution and its use at NLO:
  The coupled propagator matrix of the σ and the auxiliary fields
  Goldstone's theorem
- Curious behavior of the massive excitations in the ”hybrid-inflation” field model at $N=\infty$

The Lagrangian:

$$2L = (\partial_i \varphi_n(x))^2 - m^2 \varphi_n(x)^2 - \frac{\lambda}{12N}(\varphi_n(x)^2)^2$$
$$= (\partial_i \varphi_n(x))^2 - m^2 \varphi_n(x)^2 - \alpha(x)^2 + i\sqrt{\frac{\lambda}{3N}}\alpha(x)\varphi_n(x)^2$$

Saddle point of α and symmetry breaking shift in φ_n:

$$\alpha \to \alpha + i\sqrt{\frac{3N}{\lambda}}M^2 \qquad \varphi_1 \equiv \sigma + v\sqrt{N}, \quad \varphi_{n\neq 1} \equiv \pi_n,$$

Classical equations of motions (EOM)

$$\frac{\delta S}{\delta \sigma(x)} = -(\Box + m^2 + M^2)\sigma(x) - (m^2 + M^2)\sqrt{N}v + i\alpha\sqrt{\frac{\lambda}{3N}}(\sigma(x) + \sqrt{N}v)$$
$$\frac{\delta S}{\delta \pi_n(x)} = -\left(\Box + m^2 + M^2 + i\alpha\sqrt{\frac{\lambda}{3N}}\right)\pi_n(x)$$
$$\frac{\delta S}{\delta \alpha(x)} = -\alpha - i\sqrt{\frac{3N}{\lambda}}M^2 + \frac{i}{2}\sqrt{\frac{\lambda}{3N}}(\sigma^2 + 2\sigma\sqrt{N}v + Nv^2 + \pi_n^2).$$

Dyson-Schwinger equations up to NLO: $\frac{\delta\Gamma}{\delta\phi_U} = \frac{\delta S}{\delta\phi_U}\left[\phi_A + G_{AB}\frac{\delta}{\delta\phi_B}\right]$

$$
\begin{aligned}
\frac{\delta\Gamma}{\delta\sigma(x)} &= -(\Box + m^2 + M^2)\sigma(x) - (m^2 + M^2)\sqrt{N}v \\
&+ i\sqrt{\frac{\lambda}{3N}}(\alpha\sigma(x) + G_{\alpha\sigma}(x,x) + \alpha\sqrt{N}v) = 0, \\
\frac{\delta\Gamma}{\delta\alpha(x)} &= -\alpha - i\sqrt{\frac{3N}{\lambda}}M^2 + \frac{i}{2}\sqrt{\frac{\lambda}{3N}}(\sigma^2 + G_{\sigma\sigma}(x,x) + 2\sigma\sqrt{N}v + Nv^2 \\
&+ \pi_n^2 + (N-1)G_{\pi\pi}(x,x)) = 0.
\end{aligned}
$$

Equation of state (EoS) and gap equation:

$$
\begin{aligned}
-\sqrt{N}v\left(m^2 + M^2 - \frac{i}{Nv}\sqrt{\frac{\lambda}{3}}G_{\alpha\sigma}(x,x)\right) &= 0, \\
\frac{\lambda}{6}(v^2 + G_{\pi\pi}(x,x)) + \frac{\lambda}{6N}(G_{\sigma\sigma}(x,x) - G_{\pi\pi}(x,x)) &= M^2.
\end{aligned}
$$

DS-equations for the propagators:

$$
\begin{aligned}
iG^{-1}_{\sigma\sigma}(x,y) &= iD_0^{-1}(x,y) - \sqrt{\frac{\lambda}{3N}} G_{\alpha\phi_a}(x,z_1) G_{\sigma\phi_b}(x,z_2) \Gamma^3_{\phi_a\phi_b\sigma}(z_1,z_2,y), \\
iG^{-1}_{\sigma\alpha}(x,y) &= i\sqrt{\frac{\lambda}{3}} v\delta(x-y) - \sqrt{\frac{\lambda}{3N}} G_{\alpha\phi_a}(x,z_1) G_{\sigma\phi_b}(x,z_2) \Gamma^3_{\phi_a\phi_b\alpha}(z_1,z_2,y), \\
iG^{-1}_{\alpha\alpha}(x,y) &= -\delta(x-y) - \sqrt{\frac{\lambda}{12N}} \Big[G_{\sigma\phi_a}(x,z_1) G_{\sigma\phi_b}(x,z_2) \\
&+ G_{\pi_n\phi_a}(x,z_1) G_{\pi_n\phi_b}(x,z_1) \Big] \Gamma^3_{\phi_a\phi_b\alpha}(z_1,z_2,y),
\end{aligned}
$$

where $iD_0^{-1}(x,y) = -(\Box + m^2 + M^2)\delta(x-y)$

3-point functions at LO are classical (tree level local expressions):

$$
\Gamma^3_{\alpha\sigma\sigma}(x,y,z) = i\sqrt{\frac{\lambda}{3N}} \delta(x-y)\delta(x-z),
$$

$$
\Gamma^3_{\alpha\pi_n\pi_m}(x,y,z) = i\sqrt{\frac{\lambda}{3N}} \delta_{nm}\delta(x-y)\delta(x-z)
$$

Explicit final form

$$
\begin{aligned}
G^{-1}_{\sigma\sigma}(x,y) &= D_0^{-1}(x,y) - \frac{\lambda}{3N}\left(G_{\alpha\alpha}(x,y)G_{\sigma\sigma}(x,y) + G^2_{\sigma\alpha}(x,y)\right), \\
G^{-1}_{\sigma\alpha}(x,y) &= \sqrt{\frac{\lambda}{3N}}v\delta(x-y) - \frac{\lambda}{3N}G_{\sigma\alpha}(x,y)G_{\sigma\sigma}(x,y), \\
G^{-1}_{\alpha\alpha}(x,y) &= i\delta(x-y) - \frac{\lambda}{6N}\left(G^2_{\sigma\sigma}(x,y) + (N-1)G^2_{\pi\pi}(x,y)\right), \\
G^{-1}_{\pi\pi}(x,y) &= D_0^{-1}(x,y) - \frac{\lambda}{3N}G_{\alpha\alpha}(x,y)G_{\pi\pi}(x,y).
\end{aligned}
$$

EoS: LO $G_{\alpha\sigma}$ determines the $\mathcal{O}(1/N)$ correction:

$$
G_{\alpha\sigma}(x,x) = -\sqrt{\frac{\lambda}{3}}v\int\frac{d^4p}{(2\pi)^4}\frac{1}{\left(1+\frac{\lambda}{6}I(p)\right)(p^2-m^2-M^2)-\frac{\lambda}{3}v^2}
$$

$I(p) = i\int D_0(q)D_0(p-q)d^4q/(2\pi)^4$

Similar formulae for $G_{\sigma\sigma}$ and $G_{\alpha\alpha}$

NLO demonstration of Goldstone's theorem

EoS:

$$-\sqrt{N}v\left[m^2+M^2+\frac{i\lambda}{3N}\int\frac{d^4p}{(2\pi)^4}\frac{1}{\left(1+\frac{\lambda}{6}I(p)\right)(p^2-m^2-M^2)-\frac{\lambda}{3}v^2}\right]=0$$

Pion propagator:

$$iG^{-1}_{\pi\pi}(q)=q^2-m^2-M^2 \quad - \quad \frac{i\lambda}{3N}\int\frac{d^4p}{(2\pi)^4}\frac{1}{(q-p)^2-m^2-M^2}$$
$$\times \quad \frac{p^2-m^2-M^2}{\left(1+\frac{\lambda}{6}I(p)\right)(p^2-m^2-M^2)-\frac{\lambda}{3}v^2}.$$

At $q=0$ by EoS: $-iG_{\pi\pi}(0)=0$

Formal equations for the determination of M^2 and v^2 agree with the result of direct Dyson-Schwinger construction (see Zsolt Szép's talk)

Hybrid inflation model at large N

Φ: $O(N)$-singlet scalar coupled to $\varphi_n,\ n=1,..,N$: N-component Higgs-field:

$$L=\frac{1}{2}\partial^\mu\Phi\partial_\mu\Phi-\frac{1}{2}m_\Phi^2\Phi^2+\frac{1}{2}\partial^\mu\varphi_n\partial_\mu\varphi_n-\frac{1}{2}m^2\varphi_n^2-\frac{\lambda_1}{24N}(\varphi_n^2)^2-\frac{\lambda_2}{12N}\Phi^2\varphi_n^2.$$

"Diagonalisation" of the quartic part of the potential:

$$\lambda_1(\varphi_n^2)^2+2\lambda_2\varphi_n^2\Phi^2=\lambda_+(\psi_+^2)^2+\lambda_-(\psi_-^2)^2,$$

$$\lambda_\pm=\tfrac{1}{2}\left(\lambda_1\pm\sqrt{\lambda_1^2+4\lambda_2^2}\right),\qquad \psi_\pm^2=u_\pm\varphi_n^2+v_\pm\Phi^2,$$

$$u_+=v_-=\tfrac{\lambda_+}{\sqrt{\lambda_+^2+\lambda_2^2}},\qquad v_+=-u_-=\tfrac{\lambda_2}{\sqrt{\lambda_+^2+\lambda_2^2}}$$

Hubbard-Stratonovich transformation with two fields (α, β) :

$$L_I=-\frac{1}{2}(\alpha^2+\beta^2)+i\alpha\sqrt{\frac{\lambda_+}{12N}}(u_+\varphi_n^2+v_+\Phi^2)+i\beta\sqrt{\frac{\lambda_-}{12N}}(v_-\Phi^2+u_-\varphi_n^2)$$

$$\alpha\rightarrow\alpha+i\sqrt{3N/\lambda_+}m_+^2,\qquad \beta\rightarrow\beta+i\sqrt{3N/\lambda_-}m_-^2.$$

LO solution in the broken symmetry phase

$$m_+^2 u_+ + m_-^2 u_- + m^2 = 0,$$

$$-\sqrt{\frac{3N}{\lambda_+}} m_+^2 + \sqrt{\frac{\lambda_+ N}{12}} u_+ (v^2 + G_{\pi\pi}) = 0,$$

$$-\sqrt{\frac{3N}{\lambda_-}} m_-^2 + \sqrt{\frac{\lambda_- N}{12}} u_- (v^2 + G_{\pi\pi}) = 0$$

equivalent to the EoS:

$$m^2 + \frac{1}{6}(\lambda_+ u_+^2 + \lambda_- u_-^2)(v^2 + G_{\pi\pi}) = 0$$

The inverse propagators:

$$\begin{aligned} iG^{-1}_{\pi_n\pi_m}(x,y) &= -\delta_{nm}(\Box + m^2 + m_+^2 u_+ + m_-^2 u_-)\delta(x-y), \\ iG^{-1}_{\Phi\Phi}(x,y) &= -(\Box + m_\Phi^2 + m_+^2 v_+ + m_-^2 v_-)\delta(x-y) \end{aligned}$$

Goldstone'theorem is fulfilled

The matrix of the coupled $\sigma - \alpha - \beta$ sector

$$G^{-1}_{[\sigma,\alpha,\beta]} = \left| \begin{array}{ccc} -ip^2 & \sqrt{\frac{\lambda_+}{3}}u_+ v & \sqrt{\frac{\lambda_-}{3}}u_- v \\ \sqrt{\frac{\lambda_+}{3}}u_+ v & i(1+\lambda_+ u_+^2 I(p)/6) & 0 \\ \sqrt{\frac{\lambda_-}{3}}u_- v & 0 & i(1+\lambda_- u_-^2 I(p)/6) \end{array} \right|$$

The determinant condition for the σ mass M_σ^2:

$$\begin{aligned} & M_\sigma^2 \left(1 + \frac{\lambda_+}{6} u_+^2 I(M_\sigma)\right)\left(1 + \frac{\lambda_-}{6} u_-^2 I(M_\sigma)\right) \\ - \quad & \frac{v^2}{3}\left[\lambda_- u_-^2 \left(1 + \frac{\lambda_+}{6} u_+^2 I(M_\sigma)\right) + \lambda_+ u_+^2 \left(1 + \frac{\lambda_-}{6} u_-^2 I(M_\sigma)\right)\right] = 0 \end{aligned}$$

Non-perturbative renormalisation

for three convenient combinations of λ_1 and λ_2 and m^2:

$$\frac{1}{\lambda_\pm u_\pm^2} = \frac{1}{(\lambda_\pm u_\pm^2)_R} + \frac{1}{96\pi^2}\log\frac{\Lambda^2}{\mu^2}, \quad \frac{m^2}{\lambda_+ u_+^2 + \lambda_- u_-^2} + \frac{\Lambda^2}{96\pi^2} = \left(\frac{m^2}{\lambda_+ u_+^2 + \lambda_- u_-^2}\right)_R$$

Increased freedom to tune the mass scales of the model:

$$v^2 = \left(\frac{6m^2}{\lambda_+ u_+^2 + \lambda_- u_-^2}\right)_R ,$$

$$M_\sigma^2 = \frac{v^2}{3}\left[\frac{1}{\frac{1}{(\lambda_+ u_+^2)_R} + \frac{I_R(M_\sigma)}{6}} + \frac{1}{\frac{1}{(\lambda_- u_-^2)_R} + \frac{I_R(M_\sigma)}{6}}\right]$$

The definition of the renormalised ”inflaton” mass:

$$M_{\Phi R}^2 = m_{\Phi R}^2 + (\lambda_+ u_+ v_+ + \lambda_- u_- v_-)_R v^2$$

Case of maximal inflaton-Higgs mixing:

$$\lambda_2 \gg \lambda_1 : \qquad u_+ = -u_- = v_+ = v_- = \frac{1}{\sqrt{2}}$$

$$\lambda_+ u_+^2 \to \frac{\lambda_2}{2}, \qquad \lambda_- u_-^2 \to -\frac{\lambda_2}{2},$$

$$\lambda_+ u_+^2 + \lambda_- u_-^2 \to \lambda_1, \qquad \lambda_+ u_+ v_+ + \lambda_- u_- v_- \to \lambda_2$$

Simplified formulae for the mass scales:

$$M_\sigma^2 \approx \frac{v^2}{3}\left[\frac{I_R(p^2 = M_\sigma^2)/3}{-4/\lambda_{2R}^2 + (I_R(p^2 = M_\sigma^2)/6)^2}\right]$$

$$v^2 = \left(\frac{6m^2}{\lambda_1}\right)_R$$

$$M_\Phi^2 = m_{\Phi,R}^2 + \lambda_{2R} v^2$$

Sigma (Higgs) and inflaton scales: governed by λ_2

Condensate amplitude: governed by λ_1

Mass-spectra separated from the condensate (which would determine fermion and vector-boson masses if embedded into a gauge theory)

CONCLUSIONS (work to be done)

- NLO renormalisation, including auxiliary field renormalisation
- Treatment of $SU(N)$ symmetric Higgs model at $N \to \infty$
- Extension of the model by a scalar interacting with the Higgs fields bilinearly
- Could this be a minimal extension of SM without destroying EWPT presision tests still increasing the allowed Higgs-scale?