

Hopf algebroid twists for deformation quantization of linear Poisson structures

Stjepan Meljanac and Zoran Škoda

Abstract. In our earlier article [10] we explicitly described certain topological Hopf algebroid playing the role of the noncommutative phase space of Lie algebra type. Ping Xu has shown that every deformation quantization leads to a Drinfeld twist of the associative bialgebroid of \hbar -adic series of differential operators on a fixed Poisson manifold. In the case of linear Poisson structures, the twisted Hopf algebroid essentially coincides with our construction. Using our explicit description of the Hopf algebroid, we compute the corresponding Drinfeld twist as a product of two exponential expressions.

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1. Hopf algebroids

In this article all algebras are over a field \mathbf{k} of characteristic 0, and the unadorned tensor product \otimes is over the ground field (in deformation quantization and quantum gravity examples the field of real or complex numbers). We freely use Sweedler notation $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ for the coproducts with or without explicit summation sign.

Definition 1.1. [7, 2, 4] Given an associative algebra A , which is in this context called the **base algebra**, a **left A -bialgebroid** $(H, \mu, \alpha, \beta, \Delta, \epsilon)$ consists of

- An associative algebra (H, μ)
- Two algebra maps, the **source map** $\alpha : A \rightarrow H$ and the **target map** $\beta : A^{\text{op}} \rightarrow H$ such that $[\alpha(a), \beta(a')] = 0$ for all $a, a' \in A$. Assume in the following that on H we fix the A -bimodule structure given by $a.h.a' = \alpha(a)\beta(a')h$, $a, a' \in A$, $h \in H$.

- A -bimodule map $\Delta : H \rightarrow H \otimes_A H$ called **coproduct** satisfying coassociativity

$$(\Delta \otimes_A \text{id}_H) \otimes_A \Delta = (\text{id}_H \otimes_A \Delta) \otimes_A \Delta$$

- $\epsilon : H \rightarrow A$ called **counit** and satisfying $h_{(1)}\alpha(\epsilon(h_{(2)})) = h = h_{(2)}\beta(\epsilon(h_{(1)}))$

The following compatibilities are required for these data:

(i) Formula $\sum_\lambda h_\lambda \otimes f_\lambda \mapsto \epsilon(\sum_\lambda h_\lambda \alpha(f_\lambda))$ defines an action $\blacktriangleright : H \otimes A \rightarrow A$ which extends the left regular action $A \otimes A \rightarrow A$ along the inclusion $A \otimes A \xrightarrow{\alpha \otimes A} H \otimes A$.

(ii) The A -subbimodule $H \times_A H \subset H \otimes_A H$ (called the Takeuchi product [2]) defined by

$$H \times_A H = \left\{ \sum_i b_i \otimes b'_i \in H \otimes_A H \mid \sum_i b_i \otimes b'_i \alpha(a) = \sum_i b_i \beta(a) \otimes b'_i, \forall a \in A \right\}$$

contains the image of Δ and the corestriction $\Delta| : H \rightarrow H \times_A H$ is an algebra map with respect to the factorwise multiplication.

To see the meaning of (ii) notice that, unlike for the Takeuchi product $H \times_A H$, the factorwise multiplication is not well defined on the vector space $H \otimes_A H$ in general. Accordingly, it does not make sense to say that $\Delta : H \rightarrow H \otimes_A H$ is an algebra map. Indeed, the kernel I_A of the projection $H \otimes H \rightarrow H \otimes_A H$ of A -bimodules is the right ideal in $H \otimes H$ generated by $\beta(a) \otimes 1 - 1 \otimes \alpha(a)$, for $a \in A$, and not a two-sided ideal in general.

A Hopf algebroid is a bialgebroid with a linear map $\tau : H^{\text{op}} \rightarrow H$ called the antipode and satisfying some axioms (see e.g. [2]); several nonequivalent versions of the axioms for the antipode are used in the literature.

In this article we focus on a class of topological Hopf algebroids constructed from the Heisenberg doubles of the universal enveloping algebras of Lie algebras interpreted as noncommutative phase spaces of Lie algebra type where the universal enveloping algebra is interpreted as its coordinate sector (configuration space) and its Hopf dual as the momentum sector. This class of examples is treated in detail in [10]. The physical discussion of the coproduct for its momentum sector has been studied in many references including [1, 6]. The momentum sector there forms a (topological) Hopf algebra, while the Hopf algebroid point of view is needed only when the coproduct is extended to full noncommutative phase space; furthermore, this point of view allows for more general twists. Some other and physically important examples of Hopf algebroids are built from the data of weak Hopf algebras [2] (as those coming from symmetries in low dimensional QFTs [9]) and the study of dynamical quantum Yang-Baxter equation [5, 14].

2. Twists for Hopf algebroids

Ping Xu [14] generalized the Drinfeld twists to bialgebroids (see also [5]). Unlike in some other publications, we here use the convention of Xu about twist (elsewhere we use \mathcal{F} for his \mathcal{F}^{-1}).

Definition 2.1. [14] $\mathcal{F} \in H \otimes_A H$ is a Drinfeld twist for a left A -bialgebroid $(H, \mu, \alpha, \beta, \Delta, \epsilon)$ if the 2-cocycle condition

$$(\Delta \otimes_A \text{id})(\mathcal{F})(\mathcal{F} \otimes_A 1) = (\text{id} \otimes_A \Delta)(\mathcal{F})(1 \otimes_A \mathcal{F}) \quad (2.1)$$

and the counitality $(\epsilon \otimes_A \text{id})(\mathcal{F}) = 1_H = (\text{id} \otimes_A \epsilon)(\mathcal{F})$ hold.

We use the Sweedler-like notation for the twist $\mathcal{F} = f^1 \otimes f_1$.

Theorem 2.2. [14] *If H is a left A -bialgebroid then the formula*

$$a \star b = \mu \mathcal{F}(\blacktriangleright \otimes \blacktriangleright)(f \otimes g) = (f^1 \blacktriangleright a)(f_1 \blacktriangleright b) \quad (2.2)$$

defines an associative algebra $A_\star = (A, \star)$ structure on A with the same unit; the formulas $\alpha_{\mathcal{F}}(a) = \alpha(f^1 \blacktriangleright a)f_1$ and $\beta_{\mathcal{F}}(a) = \beta(f^1 \blacktriangleright a)f_1$ define respectively an algebra homomorphism and antihomomorphism $A_\star \rightarrow H$ turning H into an A_\star -ring; the formula

$$\Delta_{\mathcal{F}}(h) = \mathcal{F}^{-1} \Delta(h) \mathcal{F}$$

defines a map $\Delta_{\mathcal{F}} : H \rightarrow H \otimes_{A_\star} H$ which is coassociative and counital with the same counit. Moreover, $H_{\mathcal{F}} = (H, \mu, \alpha_{\mathcal{F}}, \beta_{\mathcal{F}}, \Delta_{\mathcal{F}}, \epsilon)$ is a left A_\star -bialgebroid.

Remark 2.3. (inverse cocycle) In terms of \mathcal{F}^{-1} and A_\star , we can alternatively write the cocycle condition (2.1) as

$$(\mathcal{F}^{-1} \otimes_{A_\star} 1)(\Delta \otimes_{A_\star} \text{id})(\mathcal{F}^{-1}) = (1 \otimes_{A_\star} \mathcal{F}^{-1})(\text{id} \otimes_{A_\star} \Delta)(\mathcal{F}^{-1}).$$

Remark 2.4. The proof of the associativity of \star in the theorem follows from comparing $(a \star b) \star c$ and $a \star (b \star c)$ which are from the definition (2.2) easily calculated to be

$$(a \star b) \star c = \mu(\mu \otimes \text{id})[(\Delta_0 \otimes \text{id})\mathcal{F}](\mathcal{F} \otimes \text{id})(\blacktriangleright \otimes \blacktriangleright \otimes \blacktriangleright)(a \otimes b \otimes c)$$

$$a \star (b \star c) = \mu(\text{id} \otimes \mu)[(\text{id} \otimes \Delta_0)\mathcal{F}](\text{id} \otimes \mathcal{F})(\blacktriangleright \otimes \blacktriangleright \otimes \blacktriangleright)(a \otimes b \otimes c)$$

Thus the cocycle condition (2.1) implies the coassociativity. The converse does not hold for arbitrary Hopf algebroids. Indeed, let $\mu_2 = \mu(\mu \otimes \text{id}) = \mu(\text{id} \otimes \mu)$ be the second iterate of the multiplication and \mathcal{F}_{3a} and \mathcal{F}_{3b} the left and right hand sides of the cocycle condition (2.1); we just see that for coassociativity we need to have $\mu_3(\mathcal{F}_{3a} - \mathcal{F}_{3b})(\blacktriangleright \otimes \blacktriangleright \otimes \blacktriangleright)(a \otimes b \otimes c) = 0$ for all $a, b, c \in H$. Only for some Hopf algebroids this implies that $\mathcal{F}_{3a} - \mathcal{F}_{3b} = 0$ in $H \otimes_A H \otimes_A H$. Fortunately, this is true for the undeformed Heisenberg algebra as well as for the Hopf $U(\mathfrak{g})$ -algebroid $H_{\mathfrak{g}}$ which we treat in this paper – this is the content of the Theorem 3.1.

Unlike in Xu's work, we can also treat the antipode antihomomorphism: if $\tau : H^{\text{op}} \rightarrow H$ is the invertible antipode of the Hopf A -algebroid H and \mathcal{F} a Drinfeld twist in the sense of Xu, then define $\chi^{-1} := \mu(\tau \otimes \text{id})\mathcal{F} = \tau(f^1)f_1$. This is then invertible element in H , with inverse \mathcal{F} . We claim that $\tau^{\mathcal{F}}(h) = \chi(\tau(h))\chi^{-1}$ is an antipode for the twisted A_\star -bialgebroid H^{op} .

Example. (Basic example of a noncommutative Hopf algebroid over a *commutative* base) $A = C^\infty(M)$ where M is a smooth manifold. $H = \mathcal{D}$ is the algebra of **differential operators** with smooth coefficients. Define $\Delta(D)(f, g) = D(f \cdot g)$. The base is commutative and $\alpha = \beta$ is the canonical embedding of functions into differential operators; the counit is taking the constant term. Here \blacktriangleright denotes the usual action of differential operators on functions.

Example. (related to **deformation quantization**) Ping Xu [14] extends the base algebra $C^\infty(M)$ in the above example to $C^\infty(M)[[h]]$, where h is a formal variable. Then $\mathcal{D}[[h]]$ carries a left $A[[h]]$ -bialgebroid structure obtained from A -bialgebroid \mathcal{D} essentially by extending the scalars; in this extension one works with the h -adically completed tensor product.

Theorem 2.5. [14] *If M is Poisson manifold and the formal bidifferential operator $\mathcal{F} \in \mathcal{D}[[h]]$ defines a deformation quantization of M with the star product $\mu\mathcal{F}(f \otimes g)$. Then \mathcal{F} is a Drinfeld twist for the left $C^\infty(M)[[h]]$ -bialgebroid of formal power series in regular differential operators $\mathcal{D}[[h]]$. Consequently, each deformation quantization defines also a deformation of that bialgebroid.*

We are interested in using the Hopf algebroid techniques to find explicit formulas for \mathcal{F} and also to describe the Xu's Hopf algebroid in detail in special cases.

3. Phase spaces of Lie type as Hopf algebroids

Throughout, \mathfrak{g} is a fixed Lie algebra over \mathbf{k} with basis $\hat{x}_1, \dots, \hat{x}_n$, $U(\mathfrak{g})$ is the universal enveloping algebra and $S(\mathfrak{g})$ the symmetric algebra of \mathfrak{g} . The generators of $U(\mathfrak{g})$ are also denoted $\hat{x}_1, \dots, \hat{x}_n$ (and viewed as noncommutative coordinates), but the corresponding generators of $S(\mathfrak{g})$ are x_1, \dots, x_n (and viewed as coordinates of the undeformed commutative space). The structure constants $C_{\mu\nu}^\lambda$ are given by

$$[\hat{x}_\mu, \hat{x}_\nu] = C_{\mu\nu}^\lambda \hat{x}_\lambda. \quad (3.1)$$

Let $\partial^1, \dots, \partial^n$ be the dual basis of \mathfrak{g}^* , which are also (commuting) generators of $S(\mathfrak{g}^*)$. Let $\hat{S}(\mathfrak{g}^*)$ be the formal completion of $S(\mathfrak{g}^*)$ (which should be interpreted as the algebra of formal functions on the space of deformed momenta). We introduce an auxiliary matrix $\mathcal{C} \in M_n(\hat{S}(\mathfrak{g}^*))$ with entries

$$\mathcal{C}_\beta^\alpha := C_{\beta\gamma}^\alpha \partial^\gamma \in \hat{S}(\mathfrak{g}^*), \quad (3.2)$$

where we adopted the Einstein convention of understood summation over repeated indices. In this notation introduce the matrices $\mathcal{O} := \exp(\mathcal{C}) \in M_n(\hat{S}(\mathfrak{g}^*))$ and

$$\phi := \frac{-\mathcal{C}}{e^{-\mathcal{C}} - 1} = \sum_{N=0}^{\infty} \frac{(-1)^N B_N}{N!} \mathcal{C}^N, \quad \tilde{\phi} := \frac{\mathcal{C}}{e^{\mathcal{C}} - 1} = \sum_{N=0}^{\infty} \frac{B_N}{N!} \mathcal{C}^N, \quad (3.3)$$

where B_N are the Bernoulli numbers. By a simple comparison of the expressions (3.3) we obtain

$$\tilde{\phi}_\alpha^\beta = \phi_\rho^\alpha O_\beta^\rho. \tag{3.4}$$

By \hat{A}_n denote the completion by the degree of a differential operator of the n -th Weyl algebra A_n with generators $x_1, \dots, x_n, \partial^1, \dots, \partial^n$. The underlying vector space of \hat{A}_n is thus a completion of $S(\mathfrak{g}) \otimes S(\mathfrak{g}^*)$.

Now define the elements $\hat{x}^\phi, \hat{y}^\phi \in \hat{A}_n$

$$\hat{x}_\rho^\phi := \sum_\tau x_\tau \phi_\rho^\tau, \quad \hat{y}_\rho^\phi := \sum_\tau x_\tau \tilde{\phi}_\rho^\tau. \tag{3.5}$$

Then $\hat{x}_\rho \mapsto \hat{x}_\rho^\phi$ extends to a unique algebra map $\alpha : U(\mathfrak{g}) \rightarrow \hat{A}_n$. This realization map $(-)^phi$ is related to the symmetrization (PBW) isomorphism $S(\mathfrak{g}) \cong U(\mathfrak{g})$ in the sense where other coalgebra isomorphisms $S(\mathfrak{g}) \cong U(\mathfrak{g})$ correspond to different choices of ϕ (or different ordering, see [11]). Our ϕ corresponds to the symmetric ordering [8]. Similarly, the rule $\hat{x}_\rho \mapsto \hat{y}_\rho^\phi$ extends to a unique algebra map $\beta : U(\mathfrak{g})^{\text{op}} \rightarrow \hat{A}_n$. From (3.4) it follows immediately that

$$\hat{y}_\alpha^\phi = \hat{x}_\beta^\phi O_\alpha^\beta, \tag{3.6}$$

and one can also prove (see e.g. the Appendix 1 to [10])

$$[\hat{x}_\alpha^\phi, \hat{y}_\beta^\phi] = 0. \tag{3.7}$$

The formula $\phi(\hat{x}_\alpha)(\partial^\beta) = \phi_\alpha^\beta$ defines a linear map $\phi(\hat{x}_\alpha) : \mathfrak{g} \rightarrow \hat{S}(\mathfrak{g}^*)$ which furthermore extends by the chain rule to a unique continuous derivation on $\hat{S}(\mathfrak{g}^*)$. A key property of ϕ is that the corresponding $\phi : \mathfrak{g} \rightarrow \text{Der}(\hat{S}(\mathfrak{g}^*))$ is a Lie algebra homomorphism, hence it extends to a unique right Hopf action

$$\phi : U(\mathfrak{g}) \rightarrow \text{End}^{\text{op}}(\hat{S}(\mathfrak{g}^*)).$$

This means that we can define the smash product

$$H_\mathfrak{g} = U(\mathfrak{g}) \#_\phi \hat{S}(\mathfrak{g}^*),$$

as usual: the tensor product of vector spaces $U(\mathfrak{g}) \otimes \hat{S}(\mathfrak{g}^*)$ with the multiplication $(u \otimes f)(v \otimes g) = \sum u \cdot v_{(1)} \otimes \phi(v_{(2)})(f) \cdot g$. One can check that the algebra map $H_\mathfrak{g} \rightarrow A_n$, which on elements of $U(\mathfrak{g}) \# \mathfrak{k} \hookrightarrow H_\mathfrak{g}$ agrees with the realization $\hat{x}_\alpha \mapsto \hat{x}_\alpha^\phi$ and sends $1 \otimes \partial^\beta \in \mathfrak{k} \otimes \hat{S}(\mathfrak{g}^*) \subset H_\mathfrak{g}$ to $\partial^\beta \in A_n$, is an isomorphism which will be from now on viewed as an isomorphism. We shall thus identify $\hat{x}_\mu \in U(\mathfrak{g})$ and $\hat{x}_\mu^\phi \in \hat{A}_n$ etc.

It has been shown in [10] that, with the appropriate completions implicit, $H_\mathfrak{g}$ is a Hopf $U(\mathfrak{g})$ -algebroid with coproduct Δ which on $\hat{S}(\mathfrak{g}^*) \cong U(\mathfrak{g})^*$ (identified via the symmetrization map $w_1 \cdots w_r \mapsto \frac{1}{r!} \sum_{\sigma \in \Sigma(r)} w_{\sigma_1} \cdots w_{\sigma_r}$, for all $w_i \in \mathfrak{g}$ and where $\Sigma(r)$ is the symmetric group on r letters) agrees with the transpose of the multiplication in $U(\mathfrak{g})$ and where $\Delta(u) = u \otimes 1$ for $u \in U(\mathfrak{g})$. The source and target map are α and β described above.

Notice that in the undeformed case (that is, when \mathfrak{g} is Abelian) the Hopf algebroid $H_\mathfrak{g}$ essentially coincides with (some completion of) Hopf algebroid of regular differential operators \mathcal{D} over the commutative base $C^\infty(\mathbb{R}^n)$.

Theorem 3.1. (Theorem 4 in [10]) For the topological Heisenberg double $H = H_{\mathfrak{g}} = U(\mathfrak{g}) \sharp \hat{S}(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ considered as a Hopf algebroid over $A = U(\mathfrak{g})$ the right ideal $I^{(k)}$ which is the kernel of the canonical projection $H \otimes \cdots \otimes H \rightarrow H \otimes_A \cdots \otimes_A H$ (k factors, all tensor products properly completed) coincides with the right ideal $I'^{(k)}$ consisting of all elements r in $H \otimes \cdots \otimes H$ such that

$$\mu_{k-1}(r(\blacktriangleright \otimes \cdots \otimes \blacktriangleright)(a_1 \otimes \cdots \otimes a_k)) = 0 \quad \text{for all } a_1, \dots, a_k \in H.$$

For an arbitrary Hopf algebroid one can also show that $I^{(k)} \subset I'^{(k)}$, but the converse inclusion is not present in general.

Corollary 3.2. For $H_{\mathfrak{g}}$ any \mathcal{F} such that $\Delta(h) = \mathcal{F}^{-1} \Delta_0(h) \mathcal{F} + I_{U(\mathfrak{g})}$ satisfies the cocycle condition (2.1).

Remark 3.3. (Warning on completions.) The completions in [10] are, roughly speaking, with respect to the cofiltrations on $U(\mathfrak{g})^*$ and $\hat{S}(\mathfrak{g}^*)$ induced by duality from the standard filtrations on $U(\mathfrak{g})$ and $S(\mathfrak{g})$. This is *essentially different* from using the additional formal variable h and h -adic completions as in Xu's work [14]. Naively, to fit with his work, the Lie algebra generators (or equivalently the structure constants) should be simply rescaled by the formal variable h . For many simple purposes this gives an equivalent treatment to ours. The set of series which formally converge in two variants differs however. For the main results in the present article this is important. Namely, the twists $\mathcal{F}_L, \mathcal{F}_R$ in Section 4 exist in both completions, the formulas as products of two exponentials make sense in our formalism, but these individual exponential factors do not exist in the h -adic completion, even after rescaling. Indeed, $\exp(\partial^\alpha \otimes x_\alpha)$ is not involving a small parameter (the reason is that ∂^α and x_α are dual and no rescaling could make the tensor product small) and is in fact related to an infinite-dimensional version of the canonical element, while (due cancellations in the expansion) the entire twist \mathcal{F}_L equals 1 plus series of corrections involving the small parameter. Thus unlike the exponential factors, the final results \mathcal{F}_L does exist in both formalisms and hence can be interpreted as defining a deformation quantization in the sense used in Theorem 2.5.

4. New twist

Theorem 4.1. [11, 13] In symmetric ordering, the deformed coproduct Δ on $\hat{S}(\mathfrak{g}^*) \cong U(\mathfrak{g})^*$ is given by

$$\Delta \partial^\mu = 1 \otimes \partial^\mu + \partial^\alpha \otimes [\partial^\mu, \hat{x}_\alpha] + \frac{1}{2} \partial^\alpha \partial^\beta \otimes [[\partial^\mu, \hat{x}_\alpha], \hat{x}_\beta] + \dots$$

or, in symbolic form,

$$\Delta \partial^\mu = \exp(\partial^\alpha \otimes \text{ad}(-\hat{x}_\alpha))(1 \otimes \partial^\mu) = \exp(\text{ad}(-\partial^\alpha \otimes \hat{x}_\alpha))(1 \otimes \partial^\mu).$$

The last equality follows by noting that $[\partial^\alpha, 1] = 0$.

Corollary 4.2. *In symmetric ordering, the deformed coproduct Δ on $\hat{S}(\mathfrak{g}^*) \cong U(\mathfrak{g})^*$ is also given by*

$$\Delta\partial^\mu = \exp(\text{ad}(\hat{y}_\alpha \otimes \partial^\alpha))(\partial^\mu \otimes 1)$$

Proof of the corollary. It is known [11] that the coproduct has the same symmetries as the Hausdorff formula $H(Z, W) = -H(-W, -Z)$; thus if we interchange the tensor factors in the coproduct and change the signs for every partial derivative, including in the realizations \hat{x}_α , we get the same formula but with minus sign. Changing the sign of every partial derivative in the realizations is equivalent to change of ϕ to $\tilde{\phi}$ (because partial derivatives are contracted to the structure constants so it is the same as changing the sign in structure constants), that is \hat{x}_α to \hat{y}_α . Thus

$$\Delta\partial^\mu = \partial^\mu \otimes 1 - [\partial_\mu, \hat{y}_\alpha] \otimes \partial^\alpha + \frac{1}{2}[[\partial_\mu, \hat{y}_\alpha], \hat{y}_\beta] \otimes \partial^\alpha \partial^\beta - \dots$$

We can absorb the sign $(-1)^k$ into $-\hat{y}$ -factors and this finishes the proof.

Using the Hadamard's formula $\text{Ad}(\exp(A))(B) = \exp(\text{ad } A)(B)$ we can reexpress the above formulas by

$$\Delta\partial^\mu = \exp(-\partial^\rho \otimes \hat{x}_\rho)(1 \otimes \partial^\mu) \exp(\partial^\sigma \otimes \hat{x}_\sigma) \quad (4.1)$$

$$\Delta\partial^\mu = \exp(\hat{y}_\rho \otimes \partial^\rho)(\partial^\mu \otimes 1) \exp(-\hat{y}_\sigma \otimes \partial^\sigma) \quad (4.2)$$

In particular, in the undeformed case when $C_{\mu\nu}^\lambda = 0$ and \hat{x}_α , x_α and \hat{y}_α coincide we obtain

$$\Delta_0\partial^\mu = \exp(-\partial^\alpha \otimes x_\alpha)(1 \otimes \partial^\mu) \exp(\partial^\alpha \otimes x_\alpha) \quad (4.3)$$

$$\Delta_0\partial^\mu = \exp(x_\alpha \otimes \partial^\alpha)(\partial^\mu \otimes 1) \exp(-x_\alpha \otimes \partial^\alpha) \quad (4.4)$$

Comparing the formulas for the deformed and for the undeformed case we obtain new formulas relating Δ_0 to Δ . Indeed, comparing (4.1) and (4.3) we obtain

$$\Delta(\partial^\mu) = \mathcal{F}_L^{-1} \Delta_0(\partial^\mu) \mathcal{F}_L \quad (4.5)$$

where \mathcal{F}_L is the product of the two exponentials:

$$\mathcal{F}_L = \exp(-\partial^\rho \otimes x_\rho) \exp(\partial^\sigma \otimes \hat{x}_\sigma) \quad (4.6)$$

and similarly comparing (4.2) to (4.4) we obtain

$$\Delta(\partial^\mu) = \mathcal{F}_R^{-1} \Delta_0(\partial^\mu) \mathcal{F}_R \quad (4.7)$$

where

$$\mathcal{F}_R = \exp(x_\rho \otimes \partial^\rho) \exp(-\hat{y}_\sigma \otimes \partial^\sigma) \quad (4.8)$$

The relations (4.6) and (4.8) suggest that \mathcal{F}_L and \mathcal{F}_R might be Drinfeld twists which twists the undeformed Hopf algebroid (Heisenberg algebra) to the Hopf algebroid from the Section 3. But so far we have just shown that it gives the correct formulas for $\Delta(\partial^\mu)$.

To show that \mathcal{F}_L is in fact a twist we prove analogous formulas for the rest of generators, say $\Delta(x_\mu) = \mathcal{F}_L^{-1}(x_\mu \otimes 1)\mathcal{F}_L$. Applying “inner” exponentials (3.5) and (3.6) we easily get

$$\begin{aligned} \exp(\partial^\rho \otimes x_\rho)(x_\mu \otimes 1) \exp(-\partial^\sigma \otimes x_\sigma) &= x_\mu \otimes 1 + 1 \otimes x_\mu \\ &= x_\mu \otimes 1 + 1 \otimes \hat{y}_\tau \mathcal{O}_\sigma^\tau (\phi^{-1})_\mu^\sigma. \end{aligned} \quad (4.9)$$

Now we need to apply outer exponentials to each of the two summands on the right hand side.

By induction on $k = 0, 1, 2, \dots$ one checks that (in notation (3.2))

$$\text{ad}^k(\partial^\rho \otimes \hat{x}_\rho)(1 \otimes \hat{x}_\mu) = [(-\mathcal{C})^k]_\mu^\tau \otimes \hat{x}_\tau \quad (4.10)$$

Hadamard’s formula and Eq. (4.10) imply

$$\exp(-\partial^\sigma \otimes \hat{x}_\sigma)(x_\mu \otimes 1) \exp(\partial^\rho \otimes \hat{x}_\rho) = x_\mu \otimes 1 - (\tilde{\phi}^{-1})_\mu^\tau \otimes \hat{x}_\tau. \quad (4.11)$$

For the second summand on the right hand side of (4.9), using $[\hat{x}_\sigma, \hat{y}_\tau] = 0$ and $1 \otimes \hat{y}_\tau \mathcal{O}_\sigma^\tau (\phi^{-1})_\mu^\sigma = (1 \otimes \hat{y}_\tau) \exp(\partial^\nu \otimes \hat{x}_\nu) \exp(-\partial^\lambda \otimes \hat{x}_\lambda) (1 \otimes \mathcal{O}_\sigma^\tau (\phi^{-1})_\mu^\sigma)$ we conclude that

$$\exp(-\partial^\sigma \otimes \hat{x}_\sigma)(1 \otimes \hat{y}_\tau \mathcal{O}_\mu^\tau) \exp(\partial^\rho \otimes \hat{x}_\rho) = (1 \otimes \hat{y}_\tau) \Delta(\mathcal{O}_\sigma^\tau (\phi^{-1})_\mu^\sigma) = \Delta(x_\mu), \quad (4.12)$$

where we used the known fact ([10]) that $1 \otimes \hat{y}_\tau = \Delta(\hat{y}_\tau)$. In short, we obtained the additional $x_\mu \otimes 1 - (\tilde{\phi}^{-1})_\mu^\tau \otimes \hat{x}_\tau$, but this can be shown to be in the ideal! Indeed, the formula (3.5) gives $x_\mu = \hat{x}_\sigma (\phi^{-1})_\mu^\sigma$ and (3.6) gives $\tilde{\phi}^{-1} = \mathcal{O} \phi^{-1}$, while the right ideal $I_{U(\mathfrak{g})}$ is generated by elements of the form $\hat{x}_\rho \otimes 1 - \mathcal{O}_\rho^\tau \otimes \hat{x}_\tau$ and $x_\mu \otimes 1 - (\tilde{\phi}^{-1})_\mu^\tau \otimes \hat{x}_\tau = (\hat{x}_\beta \otimes 1 - \mathcal{O}_\beta^\tau \otimes \hat{x}_\tau) ((\phi^{-1})_\alpha^\beta \otimes 1)$. It is clear here that for the twist to work it is essential that the base is larger than the field (the ideal $I_{U(\mathfrak{g})}$ is nontrivial).

Regarding that the map $H \otimes_{\mathbf{k}} H \rightarrow H \otimes_{\mathbf{k}} H$, $w \mapsto \mathcal{F}w\mathcal{F}^{-1}$ is a homomorphism of algebras our check for generators ∂^μ and x_α implies

Proposition 4.3. *For every $h \in H_{\mathfrak{g}}$,*

$$\Delta(h) = \mathcal{F}_L \Delta_0(h) \mathcal{F}_L^{-1} + I_{U(\mathfrak{g})} = \mathcal{F}_R \Delta_0(h) \mathcal{F}_R^{-1} + I_{U(\mathfrak{g})},$$

where $I_{U(\mathfrak{g})}$ is the right ideal generated by $\beta(u) \otimes 1 - 1 \otimes \alpha(u)$ for $u \in U(\mathfrak{g})$.

Finally, one proves that the undeformed right ideal I_0 generated by $x_\mu \otimes 1 - 1 \otimes x_\mu$ after twist ends in the deformed right ideal $I_{U(\mathfrak{g})}$. In fact,

Proposition 4.4.

$$\mathcal{F}_L(x_\mu \otimes 1 - 1 \otimes x_\mu) \mathcal{F}_L^{-1} \in I_{U(\mathfrak{g})}$$

Proof. We proved above that

$$\begin{aligned} \mathcal{F}_L(x_\mu \otimes 1) \mathcal{F}_L^{-1} &= \Delta(x_\mu) + x_\mu \otimes 1 - (\tilde{\phi}^{-1})_\mu^\tau \otimes \hat{x}_\tau \\ &= \Delta(x_\mu) + (\hat{x}_\beta \otimes 1 - \mathcal{O}_\beta^\tau \otimes \hat{x}_\tau) ((\phi^{-1})_\alpha^\beta \otimes 1) \\ &= \Delta(x_\mu) + I_{U(\mathfrak{g})}. \end{aligned}$$

Obviously, $\exp(\partial^\rho \otimes x_\rho)(1 \otimes x_\mu) \exp(-\partial^\sigma \otimes x_\sigma) = 1 \otimes x_\mu = 1 \otimes \hat{y}_\sigma \mathcal{O}_\rho^\sigma (\phi^{-1})_\mu^\rho$, hence by (4.12) we obtain $\mathcal{F}_L(1 \otimes x_\mu) \mathcal{F}_L^{-1} = \Delta(x_\mu)$. Subtracting the two results we obtain the assertion.

Lemma 1. [11] *In symmetric ordering, $\exp(\sum_{\alpha} t_{\alpha} x_{\alpha}) \blacktriangleright 1 = \exp(\sum_{\alpha} t_{\alpha} \hat{x}_{\alpha})$ for any formal variables t_{α} which commute with $x_{\beta}, \hat{x}_{\beta}$. In particular,*

$$\exp\left(\sum_{\alpha} \partial^{\alpha} \otimes x_{\alpha}\right)(\cdot \otimes \blacktriangleright)(1 \otimes 1) = \exp\left(\sum_{\alpha} \partial^{\alpha} \otimes \hat{x}_{\alpha}\right).$$

Theorem 4.5. \mathcal{F}_L and \mathcal{F}_R are counital Drinfeld twists for Hopf algebroid on completed Weyl algebra and by twisting they yield the Heisenberg double of the corresponding universal enveloping algebra with its canonical Hopf algebroid structure.

Proof. Both \mathcal{F}_L and \mathcal{F}_R are invertible. The proposition 4.3 and corollary 3.2 imply that the Drinfeld cocycle condition (2.1) holds. We need to show the counitality. One has to be careful when checking this as $\epsilon : H_{\mathfrak{g}} \rightarrow U(\mathfrak{g})$ is not a homomorphism. But for the symmetric ordering this is not difficult. Recall (cf. (C1)) that $\epsilon(h) = h \blacktriangleright 1$. Thus

$$(\epsilon \otimes 1)\mathcal{F}_L = \exp(-\partial^{\alpha} \otimes x_{\alpha})(\blacktriangleright 1 \otimes \cdot 1) = 1,$$

because all the higher order terms have positive power of at least some ∂^{α} -s thus yielding zero when acting upon 1. The second counitality condition is a bit more involved; using the fact that \blacktriangleright extends the regular action of $U(\mathfrak{g})$ on itself we compute

$$\begin{aligned} (1 \otimes \epsilon)\mathcal{F}_L &= \exp(-\partial^{\rho} \otimes x_{\rho}) \exp(\partial^{\sigma} \otimes \hat{x}_{\sigma})(\cdot 1 \otimes \blacktriangleright 1) \\ &= \exp(-\partial^{\rho} \otimes x_{\rho})(\text{id} \otimes \blacktriangleright) \exp(\partial^{\sigma} \otimes \hat{x}_{\sigma}) \\ &= \exp(-\partial^{\rho} \otimes x_{\rho})(\text{id} \otimes \blacktriangleright) \exp(\partial^{\sigma} \otimes x_{\sigma})(\cdot 1 \otimes \blacktriangleright 1) \\ &= (\exp(-\partial^{\rho} \otimes x_{\rho}) \exp(\partial^{\sigma} \otimes x_{\sigma}))(\cdot 1 \otimes \blacktriangleright 1) \\ &= 1 \otimes 1. \end{aligned}$$

In the third line we used the lemma 1. Similarly, one shows that \mathcal{F}_R is counital.

Corollary 4.6. $\mathcal{F}_L = \mathcal{F}_R + I_{U(\mathfrak{g})}$.

This follows from the Theorems 4.5 and 3.1.

One could of course check that not only the coproduct, but all the rest of the structure of twisted Hopf algebroid, for the twist \mathcal{F}_L is indeed our Hopf algebroid $H_{\mathfrak{g}}$. But this is rather straightforward. The representation of $H_{\mathfrak{g}}$ via a concrete twist, besides its conceptual appeal, is meant to twist many other constructions from the Heisenberg-Weyl algebra case to the case of phase space of Lie algebra type. Such applications are under investigation.

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Stjepan Meljanac

Theoretical Physics Division, Institute Rudjer Bošković, Bijenička cesta 54, P.O.Box 180, HR-10002 Zagreb, Croatia
e-mail: meljanac@irb.hr

Zoran Škoda

Faculty of Science, University of Hradec Králové, Rokitanského 62, Hradec Králové, Czech Republic
e-mail: zoran.skoda@uhk.cz