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Arbitrarily exhaustive hypergraph generation of 4-, 6-, 8-, 16-, and 32-dimensional quantum contextual sets

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Quantum contextuality turns out to be a necessary resource for universal quantum computation and important in the field of quantum information processing. It is therefore of interest both for theoretical considerations and for experimental implementation to find new types and instances of contextual sets and develop methods of their optimal generation. We present an arbitrarily exhaustive hypergraph-based generation of the most explored contextual sets [Kochen-Specker (KS) ones] in 4, 6, 8, 16, and 32 dimensions. We consider and analyze 12 KS classes and obtain numerous properties of theirs, which we then compare with the results previously obtained in the literature. We generate several thousand additional types and instances of KS sets, including all KS sets in three of the classes and the upper part of a fourth set. We make use of the McKay-Megill-Pavičić (MMP) hypergraph language, algorithms, and programs to generate KS sets strictly following their definition from the Kochen-Specker theorem. This approach proves to be particularly advantageous over the parity-proof-based ones (which prevail in the literature) since it turns out that only a very few KS sets have a parity proof (in six KS classes $<0.01\%$ and in one of them 0%). MMP hypergraph formalism enables a translation of an exponentially complex task of solving systems of nonlinear equations, describing KS vector orthogonalities, into a statistically linearly complex task of evaluating vertex states of hypergraph edges, thus exponentially speeding up the generation of KS sets and enabling us to generate billions of novel instances of them. The MMP hypergraph notation also enables us to graphically represent KS sets and to visually discern their features.

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I. INTRODUCTION

An assumed property of a classical system is that any of its measurements have values independent of other compatible measurements that might have been carried out on the system previously or counterfactually simultaneously, i.e., that the values are predetermined. The property is called the noncontextuality. This is in contrast to quantum mechanical systems whose measurements might be contextual, i.e., dependent on the context of previous or counterfactually simultaneous measurements. Such property of quantum systems is called (quantum) contextuality.

The so-called Kochen-Specker (KS) sets provide constructive proofs of quantum contextuality and therefore provide straightforward blueprints for their implementation and experimental setups. KS sets are likely to find applications in the field of quantum information, similarly to ones recently found for the Bell setups in implementing entanglements [1,2]. The assumption is supported by a recent result of Cabello [3], according to which local contextuality can be used to reveal quantum nonlocality.

Along that road, it has been most recently "demonstrate(d) that … contextuality is the source of a quantum computer's power" [4]. In particular, Howard, Wallman, Veitech, and Emerson [5] "uncover a remarkable connection between the power of quantum computers and … contextuality" [4] and prove that "contextuality is a necessary resource for universal quantum computation via magic state distillation" ([5], p. 354). ["The way of initializing the quantum bits (by means of) … superposition … is called *magic*" [4].] The scheme of Howard *et al.* [5] has been extended by Delfosse, Guerin, Bian, and Raussendorf so as to include Wigner function negativity [6].

It has also been recently shown by Raussendorf that "the measurement-based quantum computations which compute a nonlinear Boolean function with a high probability are contextual" [7]. A contextual kind of quantum gates, indispensable ingredients of quantum computational circuits, can be straightforwardly constructed from the scheme which served Waegell and Aravind to build four-dimensional (4D) complex KS sets [8].

On the other hand, Pavičić, McKay, Megill, and Fresl have shown that KS sets can serve as a generator of a new kind of lattices within Hilbert lattice representation of the Hilbert space ([9], Fig. 8), where a Hilbert lattice is an algebra underlying every Hilbert space. In addition, Megill and Pavičić have shown how new generalized orthoarguesian equations, the only known equations, apart from the orthomodularity equation itself, holding in the algebra of closed subspaces of a Hilbert space, can be generated from KS sets [10].

Another quantum information contextual KS set application is a quantum cryptography protection, as outlined by Cabello, D'Ambrosio, Nagali, and Sciarrino [11]. It has even been shown by Nagata that the KS theorem is a precondition for secure quantum key distribution (QKD) in the sense that in each QKD protocol KS noncontextuality is violated [12].

A series of KS experiments have been carried out during the last 10 years. They were implemented for 4D systems with photons [13–18], neutrons [19–21], trapped ions [22], and molecular nuclear spins in the solid states [23], for six-dimensional (6D) systems via six path possibilities for

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the photon transmission through a diffractive aperture [24,25], and for eight-dimensional (8D) systems by means of the linear transverse momentum of single photons transmitted by diffractive apertures addressed in spatial light modulators [26].

The aforementioned role of contextual sets in "supply(ing) 'magic' for quantum computation" [5] would require numerous instances of contextual sets and here KS sets as the most numerous contextual sets are likely to have an important role in designing appropriate schemes for implementations and applications. Then, in order to test different quantum gates for KS sets we should be able to engineer sufficiently large number of vectors for them, i.e., KS sets of different complexities. For constructing new algebraic structures and equations for the Hilbert space we should also have an arbitrarily increasing number of KS sets as explicitly shown in [10]. Finally, it is of theoretical significance to know the structure, features, and sizes of various KS sets. Taken together, it is important to find new classes and new instances of nonredundant nonisomorphic KS sets as well as different coordinatizations for them. It is also of importance to design algorithms and programs with the help of which we can generate an arbitrary number of different KS sets.

In this paper, we describe the discovery of large numbers (billions of them) of critical nonredundant nonisomorphic KS sets in 4-, 6-, 8-, 16-, and 32-dimensional Hilbert spaces. "Critical" means that they are minimal in the sense that a removal of any n-tuple of mutual orthogonalities, of n vectors from an n-dimensional Hilbert space, turns a KS set into a non-KS set. In other words, they represent a KS setup that has no redundancy. We describe the features of KS sets within particular KS classes which emerge as we generate the sets. We also outline patterns of distribution and generation and compare them with the other methods of generation in the literature. For instance, huge blocks of KS sets and even whole classes of KS sets turn out to be completely invisible with the latter methods.

The paper is organized as follows. In Sec. II we provide the reader with a constructive version of the KS theorem, define KS sets as well as the critical KS sets, define the parity proof for KS sets, and present the formalism, algorithms, and programs we make the use of in the paper. In Secs. III, IV, V, VI, and VI we deal with KS sets in 4D Hilbert space. In Sec. III we review the oldest KS class, the 24-24 one, which is actually a subclass of the 60-105 class we introduce in Sec. V. In Sec. IV we obtain three orders of magnitude more sets from the class 60-74 than in our previous paper [27] and 15 times more than reported in other literature (the class is also known as the 60-75 and/or 600-cell based KS class); we denoted the 60-74 class as "tentative" in the title of the section because it is particular subclass of the class 300-675 we introduce in Sec. VI. In Sec. V we elaborate on the 60-105 class defined by means of Pauli operators for two qubits in the complex Hilbert space and obtain approximately 2.5 more types of KS criticals and 3×10^4 more instances of them than known from the literature. In Sec. VI we analyze the recently discovered highly complex and extremely interwoven 300-675 class and find important subclasses at the higher end of the class. In Sec. VII we generate a class of approximately 250 000 KS criticals from the so-called Witting's master set, recently found in the literature; none of the criticals have a parity proof and therefore all the obtained sets from class are completely invisible in the standard approach via parity-proof-based algorithms and programs.

In the 6D Hilbert space, the so-called "seven context" starlike 21-7 KS critical set has recently been discovered and a challenge was issued to find bigger 6D KS sets in response to which we in Sec. VIII generate 3.7×10^6 6D KS criticals in the 236-1216 class; all but eight of the criticals lack a parity proof; we also show that the vector components of the seven-context-star KS set can be simplified and that the set itself is not contained in the latter class.

In Sec. IX we generate 10 times more types of and KS sets themselves, from the Lie algebra E8 based 120-2024 master set, than previously achieved in the literature, due to the very low number of the parity proofs (0.1%); we also construct a real starlike KS critical set and show that it is not contained in the 120-2024 class. In Sec. X we enter a sparsely charted territory of 16-dimensional (16D) four-qubit KS sets and generate approximately 2.5×10^6 more sets and approximately 70 more types of sets than known from the literature from an 80-265 master set. In Sec. XI we generate approximately 2.5×10^5 more instances and approximately 153 more types of 32-dimensional (32D) five-qubit KS criticals (from a 160-661 master set) than known from the literature. In Sec. XII we revisit the only four known three-dimensional (3D) KS criticals and show that recently spotted 13-vector set does not prove the Kochen-Specker theorem. In Sec. XIII we discuss and compare, both mutually and with those in the literature, all the KS sets we generated. In the Appendix A we give all hypergraph strings we refer to in the main body of the paper. In the Supplemental Material [28], we provide the reader with chosen KS critical sets from most of the types from all classes we considered.

II. KS SETS, MMP HYPERGRAPHS, FORMALISM, ALGORITHMS, AND PROGRAMS

Our aim is to present results in the realm of contextual setups and KS sets, methods that served us to generate them, and we introduce formalism and representation that enable us to handle them. The input and output data are extremely massive and numerous and they contain all known (from the previous literature, including our own previous papers) setups, sets, figures, hypergraphs, and diagrams as a very special and tiny portion of the ones obtained here. Before we dwell on details of the formalism we will make use of, we briefly introduce contextuality versus noncontextuality features, quote the KS theorem, and define a KS set.

The notion of noncontextuality of a system, whose observables we measure after its passing through a device, boils down to a statement that measurements of a system correspond to predetermined values of the observables during the interaction of the system with the device. A stronger statement, which is usually called the KS theorem, is that noncontextual theories assume that a predetermined result of a particular measurement of an observable of a system does not depend on measurements simultaneously carried out on other observables of the system, while quantum, contextual theories do not assume any predetermined values for outcomes of measurements, *clicks*, 0-1's, and might depend on simultaneous measurements.

Theorem 1. (*Kochen-Specker* [29–31]). In $\mathcal{H}^n$, $n \geqslant 3$, there are sets of n-tuples of mutually orthogonal vectors to which it is impossible to assign 1's and 0's in such a way that

(1) no two orthogonal vectors are both assigned the value 1;

(2) in any group of n mutually orthogonal vectors, not all of the vectors are assigned the value 0.

The sets of such vectors are called *KS sets* and the vectors themselves are called *KS vectors*.

Any KS set defined for a quantum system provides a constructive proof of the KS theorem and of the contextuality of quantum mechanics. A collection of related measurements provides an experimental verification of the theorem. Within quantum mechanics we can formalize KS set properties in the following manner. To every quantum observable of a quantum system there corresponds a linear Hermitian operator in a Hilbert space and to every state of the system associated to the observable there corresponds an eigenvector of the operator in the same space. The result of a measurement of the observable is associated with the eigenvalue of the operator. Any KS set is represented by a collection of n-tuples of mutually orthogonal (eigen)vectors from n-dimensional Hilbert spaces.

In this paper, we consider 3-, 4-, 5-, 6-, 8-, 16-, and 32-dimensional KS sets. They can be implemented in a laboratory in two different ways. By means of qubits in an n-dimensional (where $n = 2^k$, where k is a natural number $\geqslant$2) Hilbert space $\mathcal{H}^n = \mathcal{H}^2 \otimes \cdots (k) \cdots \otimes \mathcal{H}^2$ and by means of spin-$\frac{n-1}{2}$ systems. The examples of the former way are KS sets in 4D $\mathcal{H}^4$ by means of two qubits from the class 60-105 in Sec. V and from the 24-24 class [32] in Sec. III, in 8D $\mathcal{H}^8$ by means of three qubits, or in 16D and 32D spaces via four and five qubits in Secs. X and XI, respectively. The examples of the latter way are 4D 60-74 class in Sec. IV, 6D star or triangle set and the 236-1216 class in Sec. VIII, and the star or triangle set in Sec IX. In our hypergraphs approach, the calculational treatment of, and the elaboration on, all classes are the same, though. Only experimental implementations differ and we will discuss them when needed.

General formalism of n-dimensional ($n \geqslant 3;\ n \in \mathbb{N}$) KS sets and their implementation via spin-$\frac{n-1}{2}$ particles (say via, e.g., generalized Stern-Gerlach devices with a simultaneous usage of magnetic and electric fields by means of which it is possible to generate an arbitrary spin state [33]) covers any possible experimental implementation in contrast to the qubit approach which covers only n-dimensional $= 2^k$-dimensional cases ($k \in \mathbb{N}$, $k \geqslant 2$).

We represent KS sets by hypergraphs in the MMP *hypergraph* notation specified below. In a KS set, the vectors correspond to vertices of an MMP hypergraph. Vertices representing n-tuples of orthogonal eigenvectors are organized in edges of MMP hypergraphs [34].

Definition 2. MMP hypergraphs are hypergraphs in which

(i) every vertex belongs to at least one edge;

(ii) every edge contains at least three vertices;

(iii) edges that intersect each other in $n - 2$ vertices contain at least n vertices.

A KS set with n vertices and m edges is denoted as n-m. Only minimal KS sets, called *critical* KS sets, are relevant for experimental implementations since their supersets just contain additional orthogonalities that do not change the KS property of the smallest critical set.

Definition 3. KS sets that do not properly contain any KS subset, meaning that if any of its edges were removed, they would stop being KS sets, are called *critical* KS sets.

Some authors make use of a coarser notion of *(vertex-) critical* KS sets: "A KS (set) is termed critical iff it cannot be made smaller by deleting the (vertices)" [35]. However, this definition lacks operationality in identifying a huge number of critical sets which turn into a non-KS set when an edge of theirs is removed while the number of vertices remains unaltered as allowed by Definition 3. On the other hand, deleting a vertex means a removal of at least one edge.

We encode MMP hypergraphs by means of alphanumeric and other printable ASCII characters. Each vertex is denoted by one of the following characters: `1 2 ...9 A B ...Z a b ...z !` ‘‘ # $ % & ’ () * - / : ; < = > ? @ [\ ] ˆ _ ‘ { | } ˜ [32]. When all these characters are exhausted, we reuse them so as to prefix them by "+," then by "++," and so on. An example is shown in the graphical representation of a hypergraph of KS set 18-9 in the figure in Sec. III, where ASCII characters printed next to corresponding vertices from the hypergraph belong the MMP hypergraph string `1234,4567,789A,ABCD,DEFG,GHI1,29BI,35CE,68FH.` So encoded, MMP hypergraphs are generated by our algorithms and programs or introduced into our programs to be processed. Each edge is represented by a string of characters separated by commas and all of them together form a hypergraph, i.e., a KS set, as a single textual line or string which ends with a full stop. When dealing with such ASCII line encoding of MMP hypergraphs, we call them MMP hypergraphs lines or strings when needed. The order of the strings and characters is irrelevant; gaps in characters are allowed and its number is not limited; tens of thousands of them are not a problem for our programs SHORTD.C, MMPSTRIP.C, SUBGRAPH.C, VECTORFIND.C, STATES01.C, and others [9,27,32,34,36–38].

To visualize the hypergraphs, we represent them as figures showing vertices as dots and edges as straight or curved lines each connecting n-tuples of vertices. We often draw hypergraphs so as to start with the biggest loop they contain. Usually, we do not attach characters to vertices in a figure because one can always arbitrarily attach them and then use program VECTORFIND to ascribe vector components to each vertex. In chosen figures in the following sections below we show graphical representations of some of the KS sets that we found in this study in the MMP hypergraph notation.

Our standard and compact definition of MMP hypergraphs enables us to smoothly design algorithms for generation, handling, and analysis of KS sets what together amounts to MMP hypergraph language. In this work, we generate subgraphs of big chosen KS hypergraphs, which we call *master sets*, by deleting a specified number of edges from such master sets via our program MMPSTRIP. Then, we filter them on the KS property via our program STATES01 which just verifies whether they violate the conditions of the Theorem 1, i.e., whether they are KS sets. Program STATES01 carries out an exhaustive search according to a backtracking algorithm. This is a much less demanding task than a constructive upward generation we

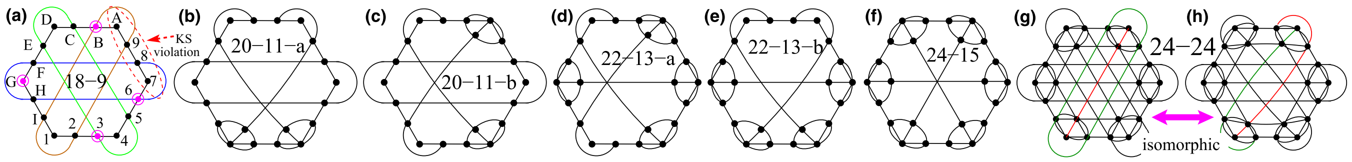


FIG. 1. Four-dimensional KS sets from the 24-24 KS class; (a)–(f) are all critical KS sets from the class; (a) and (c) were found in [45] and [47], respectively; (b), (d), and (e) were found in [34]; (f) was found in [32]; (g) and (h) are two isomorphic representations of a noncritical Peres' 24-24 KS set found in [43] [it contains all (a)–(f) as well as all the other 1226 KS sets from the 24-24 class]; (a) shows that its vertices cannot satisfy the conditions of Theorem 1; encircled vertices represent possible 1-assignments.

used previously in [34], and although we have to deal with a huge volume of hypergraphs, on computing clusters we can carry out generations successfully, as we show in subsequent sections below.

A collection of all KS subsets of a particular master set $i-j$, with i vertices and j edges, we call an $i-j$ *class* of KS sets. We can generate members of an $i-j$ class from a master set $i-j$, on a computing grid, as follows. First, we strip edges from the master set with MMPSTRIP and then filter them with SED, STATES01, SED, and SHORTD to obtain, say, $mk-l$ files, where $l=1,\dots,m$ and k depends on l in a rather involved manner depending of how many vertices, if any, were stripped together with stripped edges. Each file might contain millions of KS sets (all with l edges). Then, we use these files as input files for the next round of distributed computing to randomly generate a chosen number of nonisomorphic KS critical sets from each line, i.e., each single KS set, of the obtained files by means of STATES01 and SHORTD. Thus, we obtain arbitrary exhaustively many $p-q$ KS criticals.

In elaborating on KS sets, we, as well as other authors in the literature, make use of theory and algorithms from several disciplines: quantum mechanics, lattice theory, graph theory, and geometry. Each discipline has its own terminology which is often adopted by the authors of papers on KS sets. In this context, the terms "vertex," "atom," "ray," "1D subspace," and "vector" are synonymous, as are the terms "edge," "block," "context," and "n-tuples (of mutually orthogonal vectors)." Similarly, "MMP hypergraph" and "MMP diagram" mean the same thing.

In the literature, KS sets are very often generated and tested via the so-called *parity proof.* (The proof was generalized by Lisoněk, Raussendorf, and Singh [39].)

Definition 4. Parity proof. A parity proof of the KS theorem, in an even-dimensional Hilbert space, via a KS set is a set of k vertices of the set that form l edges (l odd) such that each vertex shares an even number of edges. Looking, e.g., at the 18-9 KS set in Fig. 1, we see that, because each vertex shares exactly two edges, there should be an even number of edges with 1's. At the same time, each edge can contain only one 1 by definition, and since there are an odd number of edges, there should also be an odd number of edges with 1's, i.e., we have a contradiction.

Parity proofs face several problems, though:

(i) KS sets with even number of edges cannot have parity proofs per definition.

(ii) Many KS sets with odd number of edges turn out not to have a parity proof, either.

(iii) In some classes of KS sets we obtained, less than 0.1% have a parity proof, and in some others, none at all.

Parity proofs are just special and particular cases of our general MMP hypergraph verification but sometimes they turn out to offer a complementary method of generation of KS sets since parity-proof-based programs are much faster than general MMP-hypergraph-based ones, when applicable.

III. TENTATIVE 24-24 CLASS OF 4D KS SETS; GENERATION OF KS SETS VIA STRIPPING OF MASTER SETS

In this section, we shall make use of the results about the 24-24 class of 4D KS sets we obtained in [32,34,40–42] to introduce the main steps and strategy we shall undertake to obtain the results in the subsequent sections. Pavičić [40] realized that one can establish a correspondence between MMP hypergraphs and systems of nonlinear equations describing mutual orthogonalities of vectors as, for instance, in the following 3D example:

$$\begin{aligned}\mathbf{x}\cdot\mathbf{y} &= x_1y_1+x_2y_2+x_3y_3 &= 0,\\ \mathbf{x}\cdot\mathbf{z} &= x_1z_1+x_2z_2+x_3z_3 &= 0,\\ \mathbf{y}\cdot\mathbf{z} &= y_1z_1+y_2z_2+y_3z_3 &= 0.\end{aligned} \tag{1}$$

The latter system is an unsolvable problem on any supercomputer, even for the smallest KS sets while ascribing 0-1 valuations, required by the definition of a KS set given in Theorem 1, to vertices of MMP hypergraphs is a problem of statistically polynomial complexity. In other words, solving for MMP hypergraphs is exponentially more computationally efficient than solving for Hilbert space vectors directly when searching for KS sets. Such a correspondence between nonlinear systems and MMP hypergraphs enables us to generate KS sets on a large scale (billions of them). This can be compared with less than a dozen of KS sets discovered by several researchers between 1967 and the end of the 20th century [30,43–48], mostly exploring highly symmetrical geometrical structures defined by mutually orthogonal vectors.

Pavičić, McKay, Merlet, and Megill [34,41] generated nonisomorphic MMP hypergraphs and filtered them by means of a program which was written for our algorithm of assigning 0's and 1's to their vertices and another algorithm for assigning vector components to vertices. The generation and assignments are exponentially complex tasks in general but applied to our KS MMP hypergraphs they turned out to be polynomially complex for the great majority of jobs. We say that they are *statistically polynomially complex*. Nevertheless, when we reached 24 vertices, the task became forbiddingly CPU-time consuming: we obtained over 300 KS sets with up to 23 vertices on a cluster with on average 100 CPUs running for several months. Among them there were only five critical KS sets. So, we started to search for another way of generating KS sets. We arrived at the idea of a faster generation as follows.

Kernaghan [45] and Cabello, Estebaranz, and García-Alcaine [47] realized that their 18-9 and 20-11 KS sets were subsets of Peres' 24-24 set [43] but, since they did not make use of graphical representation, it took them a while to find their two sets and neither they nor Peres were able to find any more KS subsets the 24-24 set (Peres even wrote a computer program for the purpose [49]).

After we generated the first few hundred KS sets in [34,41] and started to draw their hypergraphs, we visually recognized (see Fig. 1) that they were all subgraphs of the hypergraph we drew for Peres' 24-24 set. Then Pavičić, Megill, and Merlet designed an algorithm for stripping (*peeling*) edges off the latter hypergraph and obtained 1232 KS subsets [32] (including all six criticals from Fig. 1) within less than 2 min on a PC. These 1233 KS sets form a 24-24 *class* of KS sets and Peres' 24-24 set is their master set. (We would just like to mention here that we generated and scanned, during 3 CPU months, *all* nonisomorphic hypergraphs with 24 vertices and 24 edges and that among all millions of them there is only one KS set: Peres' 24-24 one.)

This led us to another aspect of generating KS sets. All vectors forming KS sets in the 24-24 class have components from the set $\{-1,0,1\}$ since Peres' 24-24 set has components from this class. However, we also found KS sets that were not subsets of the 24-24 set [e.g., the 22-11 one, shown in Fig. 3(a) of [32]] and those subsets have the components from a wider set of values (see Table 2 of [32] for the aforementioned 22-11 set). That indicated that there is another class or other classes which contain those sets or both kinds of sets and we started stripping master sets meanwhile discovered [8,38,48,50,51].

We designed algorithms and programs which exhaustively generate all KS sets from all stripped subsets of chosen master KS sets that we introduced and described in Sec. II. They are computationally rather demanding and require many CPU months of running on clusters and supercomputers but that is feasible with today's resources. In the rest of the paper, we present various outcomes of such calculations with our algorithms and the features of the critical KS sets we obtained on our clusters.

IV. TENTATIVE 60-74 CLASS OF 4D KS SETS

Waegell and Aravind have derived a 60-75 KS set from a 4D regular polytope (600-cell) with 60 pairs of vertices [50]. The vertices correspond to vectors whose components have values from the set $\mathcal{V}=\{0,\pm(\sqrt{5}-1)/2,\pm1,\pm(\sqrt{5}+1)/2,2\}$ and one can use them to write the 60-75 set ([50], Table 2). MMP hypergraph of the 60-75 generated in [38] is given in the Appendix, Sec. A 1.

Generation of smaller KS sets from the master sets will be carried out by relying on the MMP hypergraph structure only and the vertices of the obtained set can be ascribed values from $\mathcal{V}$ later on, if needed, via (a) our program VECTORFIND randomly, or (b) via our program SUBGRAPH so as to trace down vertices which survived stripping of edges. We need to ascribe values from $\mathcal{V}$ to the vertices, e.g., for an experiment (cf. [51], Fig. 1).

Megill, Fresl, Waegell, Aravind, and Pavičić [27] presented preliminary and partial results of generating subsets of the 60-75 set by stripping it of its edges and obtaining their features. Here, we present in many respects an almost exhaustive analysis of these subsets. We start by stripping just one edge at a time of the 60-75 set in 75 different ways so as to obtain 75 60-74 sets. To be more explicit, we remove one edge from the 60-75 set to get the first 60-74, then we put it back and remove another edge to the second 60-74 and so forth. It turns out that all 75 of the so-obtained 60-74 sets are isomorphic to each other and that they all reduce to a single MMP hypergraph string 60-74 given in the Appendix, Sec. A 1.

We shall therefore consider this 60-74 KS set to be a master set for all smaller KS sets we obtain from it. Therefore, we shall call the collection not a 60-75 but a 60-74 class of 4D KS sets. The number of sets from the class we generated and analyzed by running our programs over a century of CPU time on our clusters are given in Fig. 2. The stripping technique applied to the sets means a removal of one edge at the time and filtering out the KS sets with the help of several additional algorithms and programs.

In [27] we obtained only about 8000 KS sets and many were missing. Here, we have 1.54×10^9 sets and among them all types of sets that were obtained by means of much faster parity proofs and which were missing in ([27], Table 1) (denoted there by ⊗). We also obtained new types of KS sets with both even (mostly) and odd number (23) of edges that we did not obtain in [27], in particular: 38-**22**, 39-**23**, 41,43,44-**24**, 42,...,44,46,...,49-**26**, 45,50,...,52-**28**, 47,48,54,...,56-**30**, 50,56,...,60-**32**, 60-**34**.

Our aforementioned conjecture that the table in Fig. 2 shows all the types of KS criticals from the 60-74 class is based on the following statistics. The table now shows 1.54×10^9 KS criticals. The last new type, 47-30, started to appear after we reached 1.07×10^9 sets; before that, 59-32 after 5.5×10^8, 55-37 after 3.37×10^8, and all the other 150 types were already appearing within 2.15×10^8 generated sets. Here, we stress that our method of generating sets is as random as a program can possibly be and that therefore the "late" appearance of the aforementioned three types is due only to their very low occurrence among the sets, i.e., to a minuscule probability to appear at all.

This can be well illustrated by looking at the KS criticals with parity proofs. Among all 1.5×10^9 criticals only 1.2×10^5 have parity proofs and among them some are still missing. In particular, we have 3×10^6 60-39 criticals and 3.5×10^3 60-41 criticals and none of them have a parity proof although there are at least two (60-39 and 60-41 whose MMP hypergraph strings are given in the Appendix, Sec. A 1) that do have such a proof which we obtained by means of a parity-proof program in [51]. The strings are presented with their maximal loops, hexadecagon and heptadecagon (first 16 and 17 edges up to ",,,"), respectively, to facilitate graphical representation. In Fig. 3, 60-41 is drawn (vertex "2" is indicated and other vertices from the loop follow anticlockwise) and we can see that there are 22 encircled (in red online) vertices that share four edges and, of course (otherwise we would not have a

Critical 4-dim KS sets from the 60-74 class with 13 to 41 edges (columns) and 26 to 60 vertices (rows)

	13	15	17	19	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
26	1																								
30		3																							
32			1																						
33			2																						
34			5																						
36				11																					
37				9																					
38				6	11	5																			
39					31		6																		
40					42		74																		
41					22		52	131																	
42					10		126	727	238	1															
43							129	84	4636	162															
44							116	4	3270	21223	152														
45							24		871	37223	61533	66													
46							12		576	13424	284700	160088	17												
47									295	1382	169411	1488043	280073	2											
48									134	95	39418	1451894	6410844	379228	4										
49									21	2	4598	479279	9066784	19893539	185053										
50									14		940	76651	4395338	40017273	34308240	37407									
51											250	6234	975815	24774919	90756609	31426252	451								
52											67	303	120537	7296689	70507109	111015345	20902020								
53											10		7710	1076857	24336519	103127940	95148579	12010450	54						
54											12		731	92442	4476012	42608201	105755960	70800303	6603734						
55													108	3590	423982	8721957	48470624	89663708	48334576	3653990	4				
56													43	128	24845	1016418	11456719	45666317	69872507	31313317	1966077				
57															827	58353	1358794	10884172	36303582	48780264	18078665	878617			
58															103	2326	99201	1410945	8787595	25502173	29571055	8332138	270723		
59																17	2717	79484	939611	5204785	13687824	12863229	2449334	45380	
60																3	162	3905	58412	480428	2204828	4723895	3077397	347579	3519

FIG. 2. List of 1 540 184 852 nonisomorphic KS critical sets from the 60-74 class we obtained on our cluster. We conjecture that all possible types of vertex-edge sets are given here, i.e., that an exhaustive generation would not provide us with any new type. We also conjecture that an exhaustive generation might give up to about an order of magnitude more samples of these sets. We obtained no critical sets with 27, 28, 29, 31, or 35 vertices.

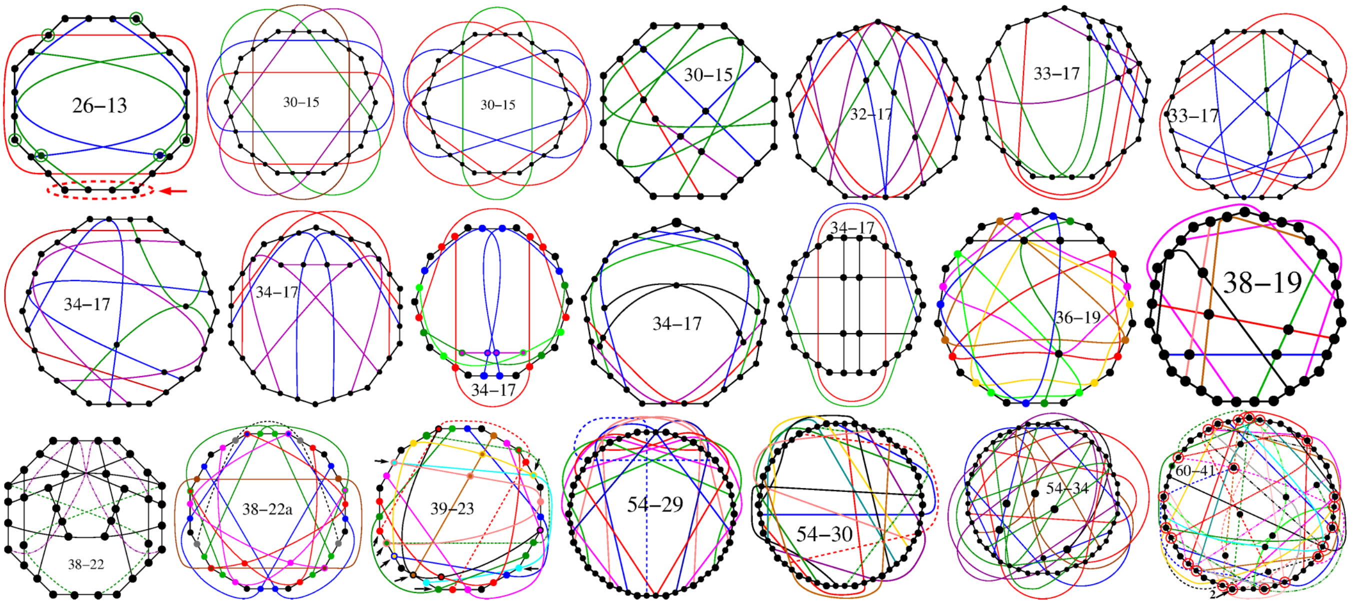


FIG. 3. MMP hypergraphs from the 60-74 class shown with the help of their maximal loops; 26-13 is the smallest set from the class: the arrow points at a "graphical proof" of contextuality [all zeros, while rings (green online) denote "1"]; 26-13 through 36-19 all have parity proofs; the first two 30-15 and the last 34-17 have two axes of symmetry; three middle 34-17, one axis; 38-22 are the smallest sets that have even number of edges; 39-23 is the smallest set with an odd number of edges which does not have a parity proof; 54-29 and 54-30 are the two smallest sets with the biggest loops (18-gon); 54-34 is a typical large set; 60-41 belongs to the largest sets of criticals; it does have a parity proof, while other 60-41 criticals do not have it (see text).

TABLE I. A sample from a complete list of 15 Pauli operator products and their eigenvectors given in Ref. [8].

Pauli product triples	Four eigenvectors of each product from the triple			
$\sigma_x^{(1)} \otimes I^{(2)}$, $I^{(1)} \otimes \sigma_z^{(2)}$, $\sigma_z^{(1)} \otimes \sigma_z^{(2)}$	$\lvert 1000\rangle$	$\lvert 0100\rangle$	$\lvert 0010\rangle$	$\lvert 0001\rangle$
$\sigma_x^{(1)} \otimes I^{(2)}$, $I^{(1)} \otimes \sigma_x^{(2)}$, $\sigma_x^{(1)} \otimes \sigma_x^{(2)}$	$\lvert 1111\rangle$	$\lvert 1\,-11\,-1\rangle$	$\lvert 11\,-1\,-1\rangle$	$\lvert 1\,-1\,-11\rangle$
$\ldots, \ldots, \ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\sigma_y^{(1)} \otimes I^{(2)}$, $I^{(1)} \otimes \sigma_z^{(2)}$, $\sigma_y^{(1)} \otimes \sigma_z^{(2)}$	$\lvert 10i0\rangle$	$\lvert 010i\rangle$	$\lvert 10i0\rangle$	$\lvert 010i\rangle$
$\ldots, \ldots, \ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\sigma_x^{(1)} \otimes \sigma_y^{(2)}$, $\sigma_y^{(1)} \otimes \sigma_x^{(2)}$, $\sigma_z^{(1)} \otimes \sigma_z^{(2)}$	$\lvert 100i\rangle$	$\lvert 01{-}i0\rangle$	$\lvert 01i0\rangle$	$\lvert 100{-}i\rangle$
$\ldots, \ldots, \ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

parity proof), not a single one of which would share three edges. The probability that a randomly generated hypergraph has such a structure is extremely low and this explains why we did not get them even after more than 10^{15} runs.

We used a procedure that strips one edge at a time of smaller and smaller sets and simultaneously checks them on KS property, KS criticality, maximal loops, number of iterations, level of classical noncontextuality of each set, etc. A choice of them is represented graphically by means of MMP hypergraphs in Fig. 3.

The KS criticals 26-13 to 36-19 all have parity proofs, and among the sets with up to 38 vertices and odd number of edges there is no one which fails the parity proof. The first sets with odd number of edges without parity proofs are 39-23 sets. One of them is shown in Fig. 3 in which arrows point to vertices that share an odd number of edges and therefore violate the parity proof condition from Definition 4. Actually, none of the 39-23 sets satisfy the parity proofs and this is the reason why this type of sets is missing in Table 1 of [51].

None of the sets with even number of edges can have a parity proof per definition. Two of the smallest such sets are 38-22 and 38-22a shown in Fig. 3. Two of the smallest sets with the biggest maximal loops in the class, octadecagon, are 54-29 and 54-30. They show an interesting property of having all vertices contained in the maximal loop like the smallest sets 26-13 and 30-15. Set 54-29 does not have a parity proof because it contains vertices that share three edges. It also has a property that some of its vertices share only one edge which most smaller sets do not possess.

As we can see from Fig. 3, the maximal loops range from octagon (26-13) to octadecagon (54-29) in contrast to the sets from the 24-24 class in Fig. 1. On the other hand, the majority of sets from the 24-24 class have edges which intersect each other at more than one vertex, while in the vast 60-74 class there is not a single such set. It follows that not only the two classes are disjoint, but that is also unlikely that they would belong to a wider class which would contain them both. However, there is a class which contains the 24-24 class, the 60-105 one, which we present in the next section.

V. 60-105 CLASS OF 4D KS SETS DEFINED BY HILBERT SPACE OPERATORS AND PROPERLY CONTAINING 24-24 CLASS

When we envisage an application of KS sets in the field of quantum computation and communication, a qubit implementation comes forward as most interesting. And, while the real vectors of the KS sets from the 24-24 class do enable a qubit representation, as recent experiments have shown, it is not clear whether the vector components of the real vectors defining the 60-74 class offer us a qubit representation. Recall that the dimension of the Hilbert space of a quantum system and the spin of this system satisfy $\dim \mathcal{H}_s = 2s + 1$. So, a 4D KS set can be realized either via an $s = \frac{3}{2}$ particle, say by means of a Stern-Gerlach device, or via two qubits: $\dim(\mathcal{H}^2 \otimes \mathcal{H}^2) = 2^2 = 4$.

In order to achieve a qubit representation in the complex 4D Hilbert space, by means of complex vectors, Aravind and Waegell [8] made use of Pauli operators (e.g., $\sigma_x^{(1)}$,$\sigma_y^{(2)}$), where the superscripts refer to one of two qubits. In a 4D Hilbert space, they form 9 mutual tensor products and 6 tensor products with the unit vectors. Altogether, these 4D operators form 15 commuting triplets each of which has four eigenvectors (tetrads) in common. There are 60 different eigenvectors that form the resulting 105 tetrads as given in Tables 1 and 2 of [8]. Their components take values from the set $\{0, \pm 1, \pm i\}$. A few lines of the former Table are given in Table I, below.

The latter table represents a 60-105 master set. Its MMP hypergraph string is given in the Appendix, Sec. A 2. By removing one of 105 edges from the string at a time, each time a different one, we obtain 105 sets. They all turn out to belong to two nonisomorphic noncritical KS sets in contrast to the 60-75 set which reduces to the unique 60-74 one. By applying the same technique as in Sec. IV, we generate critical KS sets listed in the table in Fig. 4. They make the 60-105 class of KS sets. Although the generated critical KS sets from the 60-105 class are more than two orders of magnitude less numerous than the ones from the 60-74 class, the statistics indicates that the majority of types have been generated.

MMP hypergraph of the master set 60-105 properly contains all MMP hypergraphs from the 24-24 class [8] and also the ones we obtained by means of our down-up generation in [32,34] but which did not belong to the 24-24 class as well as new ones, which do not belong to either of those two kinds, shown in Fig. 5. That is why we called 24-24 class *tentative* in the title of Sec. III. Still, with respect to vector representation, the 24-24 class is not uniquely determined by the coordinatization of the 60-105 master set. The vector components of the 60-105 set are complex (taking values from the set $\{0, \pm 1, \pm i\}$) and Peres' 24-24 master set can take over them directly as shown in the Appendix, Sec. A 2.

But, as we mentioned above, for the master set 24-24 and therefore all of its subsets there exist real coordinatizations, e.g., the one originally found by Peres, and that is what Waegell and Aravind meant when they said that "60-105

Critical 4-dim KS sets from the 60-105 class with 9 to 40 edges (columns) and 18 to 60 vertices (rows)

⊙	9	11	13	15	16	17	18	19	20	21	22	23	24	25	26	27	⊗
18	1																18
20		2															20
21		2															21
22		3	2														22
23			6														23
24			25	1													24
25			23	3													25
26			15	46													26
27				138													27
28				252		9											28
29				159	2	123											29
30				65	11	890	7										30
31						1812	215	42									31
32						1944	444	1173	1								32
33						890	381	5884	42								33
34						238	239	13776	3549	665							34
35						1	13	16501	11005	8289	1						35
							10	11606	13459	40535	411	27					36
	28	29	30		⊙		3	4059	8183	89747	9477	2162					37
44	84	1						835	3161	120118	34244	24573	1				38
45	3404	41				⊙		12	532	100316	58355	110686	1024	47			39
46	15374	1957		31	32			1	170	54486	55221	280813	11634	1648			40
47	28757	13368	342				⊙		35	15806	30340	445033	36354	19146			41
48	30880	37731	4877	77					3	2623	11214	475675	59429	92755	2085	46	42
49	22962	63876	14899	2516		33	34	⊙		66	2644	340138	60432	265314	13592	1512	43
50	12687	79656	20713	11260	741					6	641	162232	42123	521889	32795	15500	44
51	5410	73059	17174	20368	5458	206			⊙		97	39997	19078	693125	45112	60379	45
52	1775	47907	9890	22505	10815	2619	18				11	5127	5875	617069	40720	146333	46
53	394	22611	4317	18025	10589	7314	1226			⊙		41	1190	353232	26801	248373	47
54	41	7582	1577	11225	6239	8179	3970	443					182	137593	12783	297725	48
55	5	1868	457	4947	2611	5388	4534	1920	91		⊙		17	26039	3977	244506	49
56	1	706	88	1648	873	2509	2810	2676	872	11			1	2927	853	137570	50
57		1	18	497	254	864	1099	1578	1299	399		⊙		3	117	48945	51
58				1	45	241	281	612	743	495	128		⊙		17	15678	52
59					8	66	63	120	226	239	131	26		⊙	1	1773	53
60						4	11	15	29	49	40	21	2	⊙		180	54
⊗	28	29	30	31	32	33	34	35	36	37	38	39	40	⊙	⊙	⊙	⊙

FIG. 4. List of 7 720 539 nonisomorphic KS critical sets from the 60-105 class we obtained on our cluster. We conjecture that all possible types of vertex-edge sets are given here. We obtained no critical sets with 10, 12, or 14 vertices.

system contain[ed] (in 10 different ways) 24-24 systems of rays and bases used by Peres and others" [8]. The fact that the 24-24 class can have both real and complex coordinatization depends on particular structure of its sets. In contrast, the systems 21-11 shown in Fig. 5 do not possess real coordinatizations, apparently due to the δ feature of their structure (see below). Another example of different coordinatizations within the 60-105 class is Pavičić, Merlet, McKay, and Megill's 20-11a [34], shown in Fig. 1. Its 60-105 coordinatizations might be complex as given in the Appendix, Sec. A 2 as well as real. If we compared the components with those of the 24-24 set, we would see that 20-11a might be generated (stripped) directly from the 24-24. The 20-11a also possesses real coordinatizations, though, one of which is given in [34].

On the other hand, Cabello, Estebaranz, and García-Alcaine's 18-9 [47] and Kernaghan's 20-11b [45] (both shown in Fig. 1) have real coordinatizations with components from {0, ± 1} in 60-105 as given in the Appendix, Sec. A 2. By comparing their components, we can see that they are not generated directly from the presented 24-24 set with the given complex coordinatization (it does not have enough real components) but from some other subsets of 60-105. Of course, here we should pose a question as to whether there is an even wider class which properly contains the criticals from the 60-105 class, and this is an open question. Since the biggest such criticals contain only 60 vertices, such a bigger class might exist (the master set from Sec. VI has 300 vertices). However, a wider class which would properly contain both

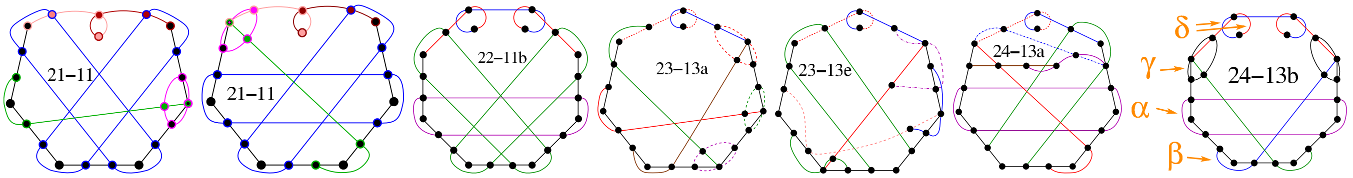


FIG. 5. MMP hypergraphs of KS critical sets from the 60-105 class with up to 24 vertices that are not isomorphic to the ones shown in Fig. 1. They share edges of the form α, β, and γ which characterize 24-24 sets but not, e.g., the one of the form δ, which is specific to the 60-105 sets. Maximal loops of the criticals shown here range from hexagons to octagons.

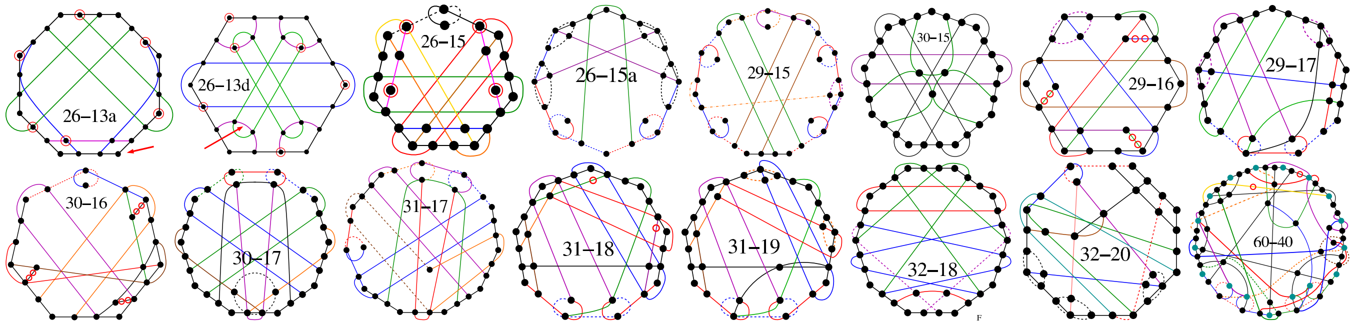


FIG. 6. Bigger 60-105 class criticals; sets with even number of edges (no parity) compared with sets with odd number of edges (with parity) of the same vertex size; arrows indicate edges at which conditions (1) and (2) of the KS theorem are violated and the theorem proved; rings (red online) in 29-16, 30-16, 31-18, and 60-40 denote vertices that share just one edge; 26-15 is one of the smallest sets without parity proofs; 60-40 is one of the biggest criticals; its maximal loop forms a heptadecagon (17-gon).

60-74 and 60-105 might not exist since these two classes have too disparate properties. First, not a single critical KS set from the 60-105 class is isomorphic to any of 1.5×10^9 critical KS sets from the 60-74 class. Second, there is an important structural difference together with all similarities.

The similarities are of the α, β, and γ kind shown at 24-13b and 24-13c in Fig. 5. α is an edge whose vertices each share a single edge from the maximal loop; β consists of two such vertices, one which shares two loop edges and a third edge and one which shares only that third age; γ is the third edge from the previous β definition.

A definite difference with and a dominant feature of 65-105 sets is the δ feature (see Fig. 5). It refers to two neighboring edges from the maximal loop exclusively sharing two vertices, i.e., intersecting each other at two vertices which do not share any third edge. The δ feature characterizes most of the criticals shown in Figs. 5, 6, and 7. It might correspond to a rank-2 projector and be related to the fact that in a KS test one need not distinguish which of the two vertices that share two edges was assigned a 1. The role of projectors of a higher rank in a description of KS sets has been explored by Waegell and Aravind in details in [8].

The portion of sets from the 60-105 class with an odd number of edges which possess the parity proofs and the overall number of sets from the class with the parity proofs is much higher than in the 60-74 class. Of 7.5×10^6 60-105 criticals, we obtained, 5.72×10^6 have parity proofs, i.e., 76.3%. The latter number includes six criticals from the former 24-24 class which all have parity proofs. There are 132 types of KS criticals with an odd number of edges of which 45 were previously reported by Waegell and Aravind [8] and additional 12 by Pavičić [52] and 111 with an even number of edges of which 22 were previously found by Pavičić [52].

A general feature of all classes is that smaller sets have only odd number of edges and that they all have parity proofs. On the other hand, among large sets with odd number of edges there are only a very few ones with the parity proofs. As we saw in Sec. IV, we did not obtain a single such 60-39 or 60-41 set in the 60-74 class although they exist (and are given above) and of 21 60-39 sets in the 60-105 class no one has a parity proof and, to our knowledge, it is not known whether such a set exists. One of 11 smallest sets without a parity proof is the 26-15 shown in Fig. 6. Encircled vertices (in red online) do not satisfy the parity-proof condition; they do not share an even number of edges.

The smallest sets with even number of edges are 29-16. In Fig. 6, a sample of them is shown with vertices which share only one edge drawn as rings (red online). As the number of vertices and edges increase, there are fewer and fewer such vertices which are dominant among 3D KS criticals (see Sec. XII). Yet, there is one of them in the 60-40 set. Our generation of KS sets via stripping of master sets was so far completely random. As we already stressed, this does require a considerable amount of CPU time. What slows down the generation is not the stripping itself, which is extremely fast, but filtering on the KS property and criticality. Algorithms which would be focused on particular arrangement of vertices and edges might prove more efficient and even serve us to obtain KS sets without previous stripping from any master set. Possible arrangements of such a kind are the ones which would have all vertices contained within a single loop as 26-13 to 46-23 or nearly so as 50-25 and 54-27 in Fig. 7.

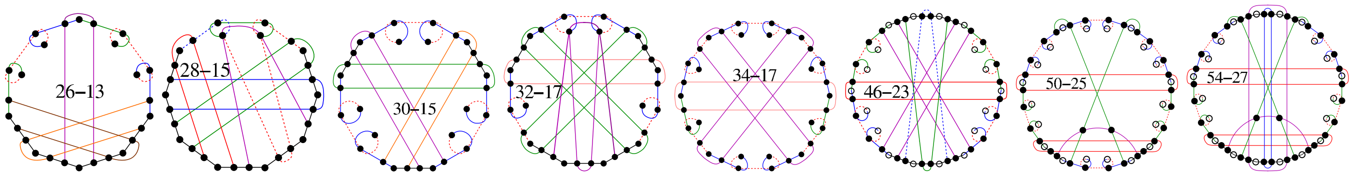


FIG. 7. 26-13 to 46-23 samples of 60-105 criticals with all the vertices contained in the maximal loop; there are no such sets with 50 or more vertices; 50-25 and 54-27 are the closest structures; rings in 46-23 to 54-27 denote the end vertices of edges.

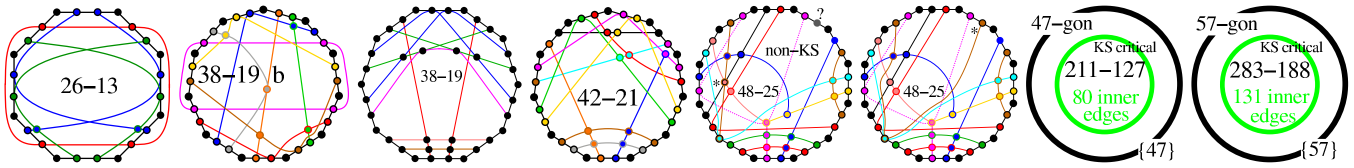


FIG. 8. KS criticals from the 300-675 class together with one non-KS set (see text); KS criticals from the higher vertex-edge group (211 to 283 vertices and 127 to 188 edges) are represented by circles since the vertices and edges are too numerous to be discernible in a figure (they are all listed in Fig. 9, though); black circles represent maximal loops with 47 (Schläfli symbol {47}) and 57 edges ({57}).

VI. 300-675 CLASS OF 4D KS SETS CONTAINING 60-74 CLASS

Waegell, Aravind, Megill, and Pavičić made use of 600-cell convex regular 4-polytope to obtain a 60-75 master KS set and a huge number of KS criticals which we call the 60-74 KS class [51]. Three years later, Waegell and Aravind considered its dual 120-cell and obtained a 300-675 master set and from it a number of different KS sets via parity proofs [53]. In particular, using parity-proof algorithms and programs, they found the following 102 types of critical KS sets from their 300-675 master sets: 38-**19**, 42-**21**, 44 . . . 46-**23**, 48 . . . 50-**25**, 50 . . . 54-**27**, 52 . . . 58-**29**, 54 . . . 62-**31**, 56 . . . 66-**33**, 58 . . . 70-**35**, 60 . . . 74-**37**, 53 . . . 78-**39**, and 65 . . . 82-**41**. We show MMP hypergraphs for some of them (38-19, 42-21, 48-25) in Fig. 8.

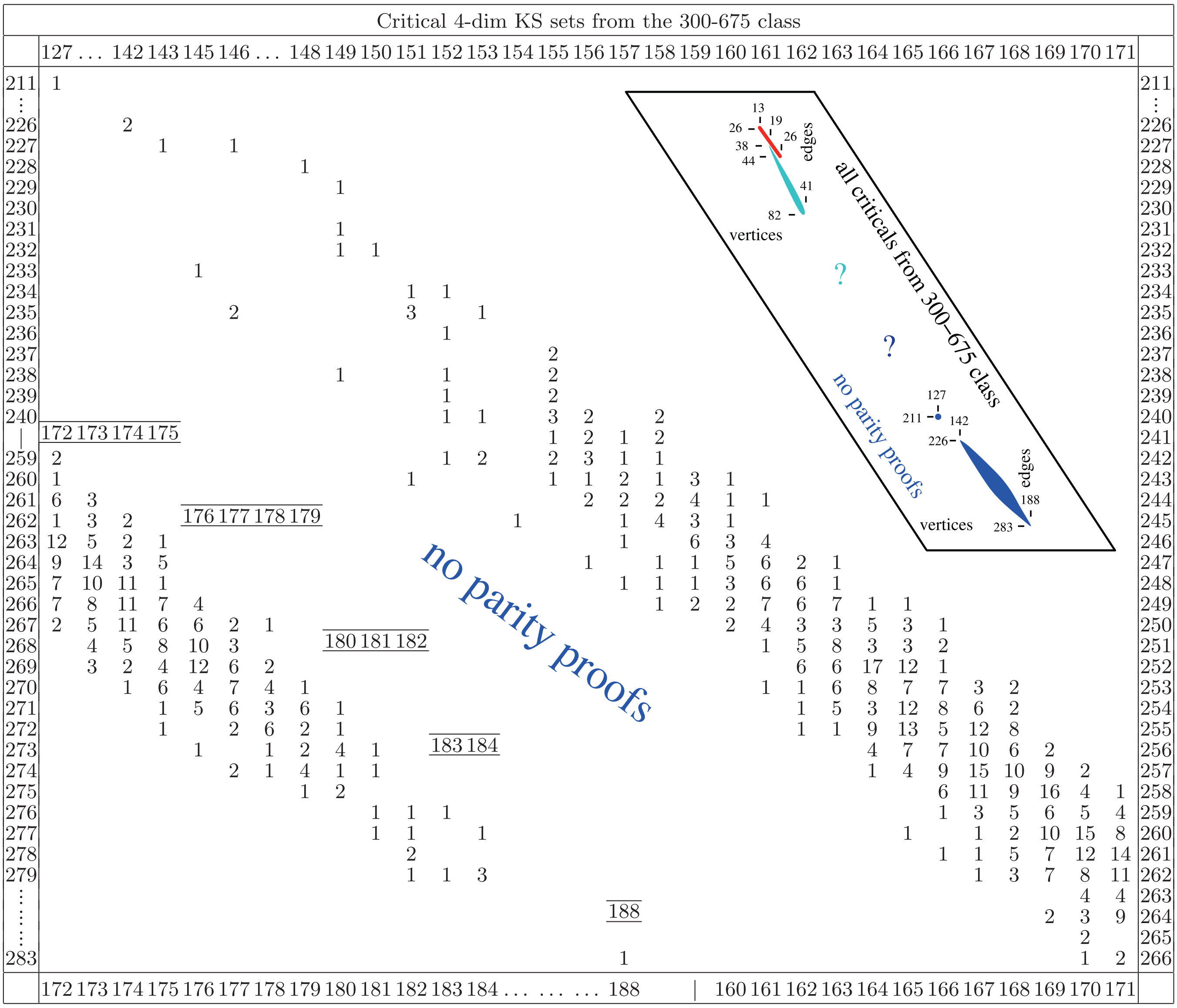
Among the smallest KS criticals we generated from the 300-675 master set are one 26-13 (shown in Fig. 8), two (nonisomorphic) 30-15, one 32-17, one 33-17, four 34-17, two 38-19 (one of them, 38-19b, is shown in Fig. 8), one 43-24, and one 44-26. Apart from the last two, all of them have parity proofs. These KS criticals with parity proofs are subgraphs of the master set 60-74 and therefore belong to the 60-74 class. Waegell and Aravind actually show in [53], by the very construction of the 300-675 master set, that the 60-75 master set is properly contained in it, i.e., that the hypergraph of the 60-75 master sets is contained in the hypergraph of the 300-675 master set.

Next, in Fig. 8, we present three hypergraph MMP representations of the KS criticals obtained in [53]: 38-19 (max loop: 10-gon), 42-21 (11-gon), and 48-25 (12-gon). Their MMP hypergraphs are given in the Appendix, Sec. A 3. Our program SUBGRAPH shows that these KS criticals are not subgraphs of the master set 60-74 (or 60-75) and that, therefore, the class 60-74 does not contain them. Here, we use the opportunity to show yet another advantage of the hypergraph approach to KS sets. Waegell and Aravind made a misprint somewhere in their Table 7 [53] which should have defined their 48-25 set but an automated translation gives a hypergraph denoted “non-KS 42-21” in Fig. 8. To find the misprint in their list of vertices and edges, one should invest a considerable amount of time and most likely they themselves as well. However, in our representation it is immediately visually apparent that in a parity proof the vertex can neither share three edges, denoted by “*”, nor just one, denoted by “?”. Therefore, we can easily amend the misprint by disconnecting the brown edge from the *-vertex and extending it to the ?-vertex, so as to obtain the 48-25 KS critical set shown as the next set in the figure; its ASCII MMP representation is given above. It provably does not belong to the 60-74 class.

Further advantages are obvious from the generation of a cluster of unprecedentedly big KS criticals indicated in the last two figures in Fig. 8 and listed in all details in Fig. 9. The generation of KS criticals in the 300-675 class is an extremely demanding task due to the intricacy of the master set 300-675 itself which stems from the high number of vertices. If we strip too many loops in the first step with MMPSTRIP, we shall find ourselves in the non-KS desert, i.e., the probability of finding a KS set will be too small. If we strip only, say, 500 loops, from the master set, verification of whether a single obtained MMP hypergraph is a KS set and if it is to reduce it to a critical KS set will take between one and three CPU months (3 GHz). At the first glance, it might look as if Waegell and Aravind also stumbled upon this problem of intricately interwoven edges and orthogonalities: “we have not found any (set) with more than 41 bases, but we cannot be sure about the upper limit because our searches have been limited to only the reduced sets in Table 4” ([53], p. 1093, bottom).

However, they actually could not have found them because with their parity-proof-based programs they could not have seen them at all. More precisely, 14% of all criticals in the 300-675 class have a parity proof but the probability KS sets having them is not uniformly distributed throughout the class. All KS sets with parity proofs are in the bottom part of the class. KS sets from the top part of the class do not have parity proofs; none of the 221-127 to 283-188 generated KS criticals have a parity proof. Hence, they are invisible for parity-proof-based algorithms and programs and since search algorithms in the literature rely almost exclusively on parity proofs we give an MMP representation of the 221-127 critical KS set (47-gon) in the Appendix, Sec. A 3.

Higher criticals from the class 300-675 we obtained and presented in Fig. 9 are far less numerous than KS criticals from any other class we presented in this paper. This is, however, not due to a small number of sets in the class: their overall number is according to our tests staggeringly huge; this is due to the fact that their generation is computationally extremely demanding and time consuming. Therefore, we generated the sets in stages and subjected them to several levels of filtering. We first randomly stripped 400 to 550 edges from the master set 300-675 by MMPSTRIP thus obtaining 150 groups of sets with 275 down to 125 edges. Then, we filtered these sets for the KS property and randomly reduced them to criticals by means of STATES01. This procedure takes up to three CPU months for each single critical. We generated higher criticals from the KS noncritical sets in the range from 190 to 275 edges. For example, the 211-127 KS critical we obtained from a set with 190 edges. For sets with less than 190 edges we observed a sudden drop to criticals with up to 40 edges.



Critical 4-dim KS sets from the 300-675 class

	127	. . .	142	143	145	146	. . .	148	149	150	151	152	153	154
211	1													
⋮														
226			2											
227				1		1								
228								1						
229									1					
230														
231									1					
232									1	1				
233					1									
234											1	1		
235						2					3		1	
236												1		
237														
238									1			1		
239												1		
240												1	1	
241														
242												1	2	
243											1			
244														
245														1

	155	156	157	158	159	160	161	162	163	164	165	166	167	168	169	170	171
237	2																
238	2																
239	2																
240	3	2		2													
241	1	2	1	2													
242	2	3	1	1													
243	1	1	2	1	3	1											
244		2	2	2	4	1	1										
245			1	4	3	1											
246			1		6	3	4										
247		1		1	1	5	6	2	1								
248			1	1	1	3	6	6	1								
249				1	2	2	7	6	7	1	1						
250						2	4	3	3	5	3	1					
251							1	5	8	3	3	2					
252								6	6	17	12	1					
253							1	1	6	8	7	7	3	2			
254								1	5	3	12	8	6	2			
255								1	1	9	13	5	12	8			
256										4	7	7	10	6	2		
257										1	4	9	15	10	9	2	
258												6	11	9	16	4	1
259												1	3	5	6	5	4
260											1		1	2	10	15	8
261												1	1	5	7	12	14
262													1	3	7	8	11
263																4	4
264															2	3	9
265																2	
266																1	2

	172	173	174	175	176	177	178	179	180	181	182	183	184	. . .	188
259	2														
260	1														
261	6	3													
262	1	3	2												
263	12	5	2	1											
264	9	14	3	5											
265	7	10	11	1											
266	7	8	11	7	4										
267	2	5	11	6	6	2	1								
268		4	5	8	10	3									
269		3	2	4	12	6	2								
270			1	6	4	7	4	1							
271				1	5	6	3	6	1						
272				1		2	6	2	1						
273					1		1	2	4	1					
274						2	1	4	1	1					
275								1	2						
276										1	1	1			
277										1	1		1		
278											2				
279											1	1	3		
⋮															
283															1

FIG. 9. Critical 4D KS sets from the 300-675 class; the very table shows only big KS criticals with 127 to 188 edges (columns) and 211 to 283 vertices (rows); the inset shows all critical KS sets from the 300-675 class: at the bottom right (blue online) are all criticals from the outer table, at the top left, upper line (red online), are 26-13, . . . , 44-26 we obtained (belonging to the 60-74 class as well; not shown in the table), and also at the top left, lower line (cyan online), are 28-19, . . . , 82-41 obtained by Waegell and Aravind [53] (equally not shown in the table itself).

VII. 148-265 CLASS OF 4D KS SETS

Waegell and Aravind [54] showed that the Penrose dodecahedron, Zimba and Penrose used to construct their 40-40 non-critical KS set [31], can be extracted from the *Witting polytope* in $\mathbb{C}^4$. Actually, Waegell and Aravind consider a 148-265 KS master set and its subsets, 40-40 being one of them. Since this is a work in progress, we shall not go into details but will only list the types of KS criticals the master set 148-265 can be reduced to and give two of their hypergraphs, in Fig. 10, so as to round up our presentation of generation of KS criticals from all known 4D KS master sets and their classes.

Waegell and Aravind in [54] make use of a rather involved coordinatization, but they also indicate that a simpler one, in which vector components take the values from the set $\{0,\pm 1,\pm\omega,\pm\omega^2\}$, where $\omega = e^{2\pi i/3} = (-1+i\sqrt{3})/2$, can be used ([54], Eq. (6)). We explicitly verified that, in the master set 148-265, vectors can indeed be ascribed a valuation from this set which means that all sets from the 148-265 class can easily be given a random valuation with the help of our program VECTORFIND by simply introducing the seven values given above as its options. Two examples of such a valuation are 40-23 and 49-27 KS criticals. 40-23 MMP hypergraph, shown in Fig. 10(a), is one of 56 40-23 subgraphs of Penrose's 40-40 KS hypergraph. 49-27, shown in Fig. 10(b), is the smallest critical from the 148-265 class which is not contained in its 40-40 set. MMP hypergraph strings of 40-23 and 49-27 are given in the Appendix, Sec. A 4.

In contrast to the smallest sets from the other 4D KS classes, the above smallest hypergraphs do not show geometrical symmetries and that is caused by the geometrical features of the Witting polytope which in turn cause that the vertices

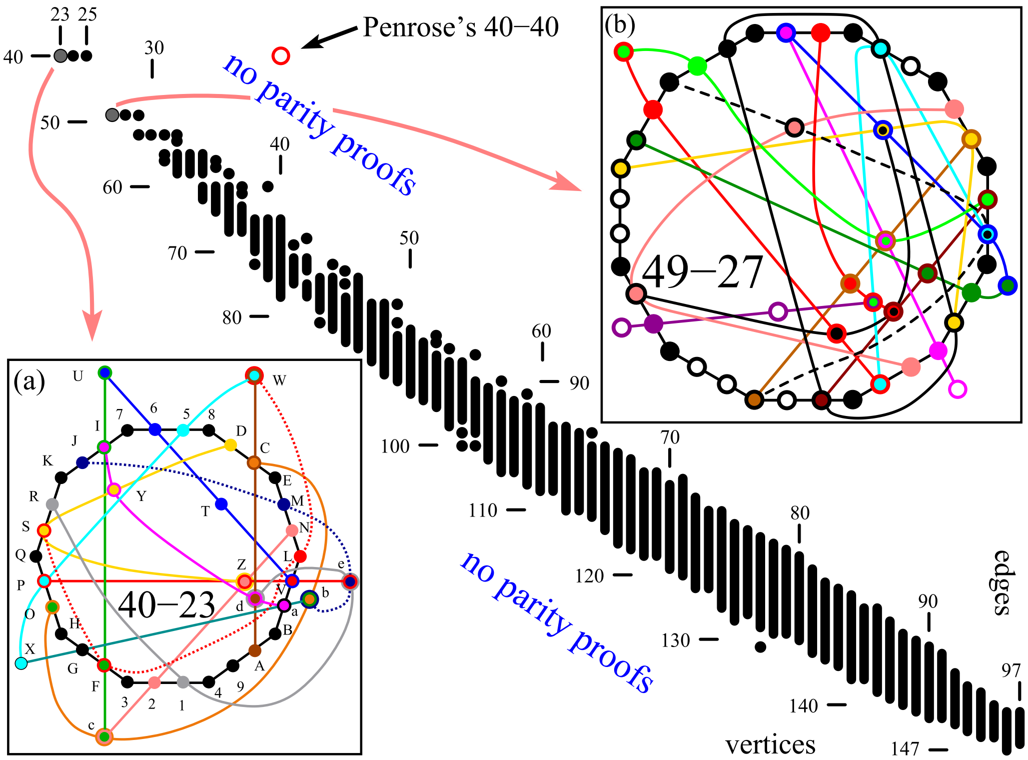


FIG. 10. KS criticals from the 4D 148-265 class; none of them have a parity proof; 40-40 circle (red online) indicates the Penrose 40-40 noncritical KS set; inset (a) shows one of smallest KS criticals generated from Penrose's 40-40 set; inset (b) shows one of the smallest KS criticals not contained in the 40-40 set.

share both even and odd number of edges, i.e., that they do not have parity proofs. Actually, none of 250 140 KS criticals in the 148-265 class we obtained have a parity proof, so they are completely invisible for the parity-proof-based algorithms and programs.

The maximal loops of the criticals are up to 36-gons big and therefore smaller than the ones of all the higher KS criticals from the 300-675 class but bigger than ones of all the other KS criticals from any other class. Similarly to 60-74 and 300-675 and unlike 24-24 and 60-105 classes, no two edges share more than one vertex. The program SUBGRAPH verified that the master set 148-265 is not a subgraph of the master set 300-675. Program SHORTD verifies that the classes 148-265 and 300-675 are completely disjoint.

VIII. ⋆/△ 21-7 6D KS SET AND 236-1216 CLASS OF 6D KS SETS

Lisoněk, Badziag, Portillo, and Cabello [24] recently found a 6D 21-7 KS set which they drew in the form of a seven-pointed star, a regular heptagram with Schläfli symbol $\{7/3\}$, as shown in Fig. 11. (It was experimentally implemented in [25].) They chose vector component values from the set $\{0,1,\omega,\omega^2\}$, as we did in Sec. VII, however, since ω is a cube root of 1 and therefore $\omega\times\omega^2=1$, and 1 is already present as a component, for their set ω^2 is not needed. To see this, we start with the MMP hypergraph representation of the seven-star set: `123456,6789AB,BCD3EF,F5G8HI,IAJD2K,KE4G7L,LH9JC1`. We assign `1` to any point and then proceed along the edges. Our program VECTORFIND can then ascribe the components to vertices: `1`$=(0,0,0,0,0,1)$, `2`$=(0,0,0,0,1,0)$, …, `6`$=(1,0,0,0,0,0)$, `7`$=(0,1,0,\omega,1,\omega)$, `8`$=(0,0,1,1,\omega,\omega)$, `9`$=(0,\omega,\omega,1,1,0)$, `A`$=(0,\omega,1,\omega,0,1)$, `B`$=(0,1,\omega,0,\omega,1)$, `C`$=(1,\omega,1,0,\omega,0)$, `D`$=(\omega,1,1,0,0,\omega)$, `E`$=(\omega,\omega,0,0,1,1)$, `F`$=(1,0,\omega,0,1,\omega)$, `G`$=(1,0,0,\omega,\omega,1)$, `H`$=(\omega,0,1,\omega,1,0)$, `I`$=(\omega,0,\omega,1,0,1)$, `J`$=(1,1,\omega,\omega,0,0)$, `K`$=(1,\omega,0,1,0,\omega)$, `L`$=(\omega,1,0,1,\omega,0)$. Recall that dot products (orthogonality of vectors) involve the complex conjugates, e.g., `K`$\cdot$`L`$^\dagger=\omega^*+\omega+0+0+1+0+0=(-1-i\sqrt{3})/2+(-1+i\sqrt{3})/2+1=0$.

The heptagram is isomorphic to a triangle (△) hypergraph shown in Fig. 11 below the star (⋆). The advantage of the triangular representation is that it can describe both odd- and even-dimensional sets while the starlike representation is limited to the even-dimensional sets. 4D triangle (five-pointed star), which does not admit a 0-1 state, would be a 10-5 KS set if it had a vectorial representation in the complex Hilbert space, but apparently it does not have it. Here, we stress that all the results we obtained for the KS sets of the 60-105 class depend on vector components from $\{0,\pm 1,\pm i\}$. There are 4D KS sets with vector components from other sets, e.g., the ones from the 60-74 class. There are also hypergraphs which do not admit noncontextual 0-1 states, e.g., those smaller than the 18-9 [34], or the above 10-5 one, for which we actually do not know whether they have a vectorial representation with vector component values from some other sets. A direct solving of nonlinear equations which would answer this question is rather demanding.

Neither the 5D triangle (15-6 set) nor the 7D triangle (28-8) are KS hypergraphs (they do admit 0-1 states), but the 8D

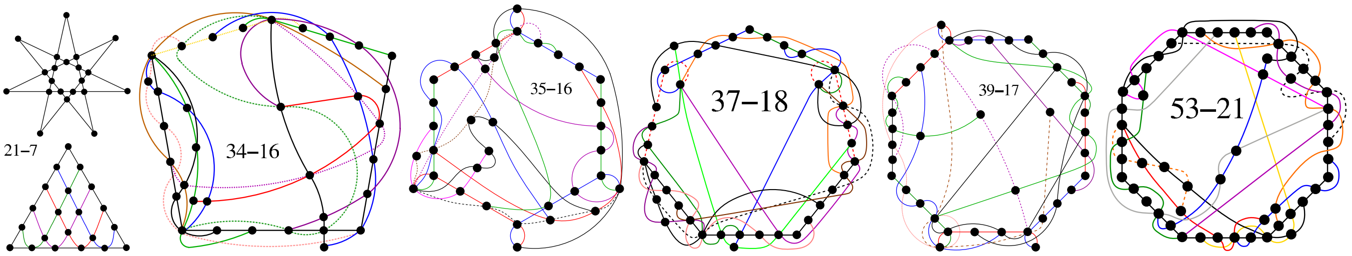


FIG. 11. 6D KS critical sets: $\star/\triangle$ 21-7 and KS from the 236-1216 KS class; $\star$ 21-7 is from [24]; $\triangle$ 21-7 is isomorphic to $\star$; others are critical KS sets from the 236-1216 class; 53-21 has a parity proof.

one (36-9) is. The latter KS set is also not a subgraph of the 120-2024 class (see Sec. IX). The authors of [24] have made an attempt to find a bigger 6D set but did not find any.

Waegell and Aravind appreciated the approach as the first one "in a dimension that is not of the form 2^N" [53], meaning that the 6D space cannot "host" qubits (recall that two qubits reside in the 2^2D, i.e., 4D Hilbert space, three qubits in the 2^3D, i.e., 8D space, etc.). Subsequently, Aravind and Waegell [55] designed a 6D 236-1216 master KS set but since it did not allow parity proofs they could not generate smaller sets with their parity-proof programs. So, they sent the master set to us and we generated 3.7×10^6 KS criticals in this paper. We say that they make the 236-1216 class. Their statistics is shown in Fig. 12. The vector components take values from the set $\{0,\pm 1/2,\pm 1/\sqrt{3},\pm 1/\sqrt{2},1\}$. The class does not contain the 21-7 KS set, though (verified with SUBGRAPH).

Aravind [55] has arrived at the 236-1216 master KS set by considering hypercubes which led him to a hexeract (6-cube, 6D cube) with Schläfli symbol {4,3,3,3,3} or $\{4,3^4\}$. The master set written in the MMP notation occupies more than three pages, so we do not print it here. The approach of Aravind and Waegell is very geometrical and unorthodox and by no means straightforward, so, it is outside of the scope of this paper. It will be presented in detail in a separate publication. The master set in the MMP notation is given in our repository http://goo.gl/xbx8U2.

The features of the 236-1216 class are as follows:

(1) Its KS sets cannot be implemented via qubits but can via spin-$\frac{5}{2}$ quantum systems.

(2) Its smallest KS sets have an even number of edges and small sets with odd and even number of edges are evenly distributed, unlike in any other class.

(3) Although the number of vertices of the master set is comparable with the 4D 300-675 and the number of edges is twice as high, the criticals are computationally much easier to generate; a generation of a single KS critical is up to 1000 times faster.

(4) Types of sets with a definite number of edges and different number of vertices are more numerous than in other classes (columns in Fig. 12 are higher than in other tables); a dynamic algorithm compensated for a lower occurrence of smaller KS sets.

(5) Edges connect vertices in much more irregular way than in other classes as the figures in Fig. 11 show. We were not able to find a single symmetric hypergraph.

(6) Statistics from Fig. 12 shows gaps in the KS sets with high number of edges indicating that a more extensive generation would generate many more sets possibly with higher number of edges and vertices.

There is another peculiarity we should mention. As already stressed above, in the literature, most of the KS proofs have been found via parity proofs. However, in the 236-1216 class among 3.7×10^6 KS critical sets we generated we found only eight KS criticals with a parity proof. Their edges are in the interval from 21 to 39. We shall present and discuss them in detail in a subsequent publication, and here we only show one of them (53-21) in Fig. 11.

IX. 120-2024 CLASS OF 8D KS SETS AND $\star/\triangle$ 36-9 8D KS SET

We start with a brief history of generation of 8D KS sets which can be realized with either three qubits ($2^3=8$) or spin-$\frac{7}{2}$ systems. Kernaghan and Peres produced a 36-11 KS critical set and a 40-25 noncritical one (experimentally implemented in [26]) from which several smaller ones including 36-11 can be obtained [44]; Ruuge and van Oystaeyen gave a scheme for constructing 8D KS proofs but did not themselves construct any [56]; Ruuge claimed to have given an example of a 36-vertex 8D KS set [35] but we were not able to identify its octads of orthogonal vertices in [35] (nor to contact him), so, we could not verify whether it is isomorphic to 36-11 from [44] (as claimed in [35]); and, finally, Planat discussed 8D KS sets that can be obtained from the Kernaghan-Peres' 40-25 KS set [57]. Waegell and Aravind obtained a KS master set with 120 vertices and 2025 edges and, from it, many smaller 8D KS sets, including noncritical Kernaghan-Peres' 40-25 one [58] (see also [59]). In this paper we generate 6.9×10^6 nonisomorphic KS criticals, listed in the table in Fig. 13, from Waegell-Aravind's 120-2025 master set. We also produce a new star or triangle ($\star/\triangle$) 36-9 KS set which is not a subgraph of the 120-2025 master set.

To obtain KS sets, in Refs. [56,58], the authors made use of the Lie algebra E8. Waegell and Aravind reduced it to a collection of 120 vectors (rays, vertices) and 2025 bases (octads, edges) [58] to obtain their 120-2025 KS master set. We verified that by peeling off one edge at a time, we obtain 2025 varieties of the 120-2024 KS sets which are all isomorphic to each other and therefore reduce to a single 120-2024 KS master set from which we generate the 120-2024 KS class, i.e., smaller KS critical sets. Critical KS sets from the 120-2024 class are given in the table in Fig. 13.

The coordinatization (vector components) in [58] is taken over from Richter and is based on tetrads formed by

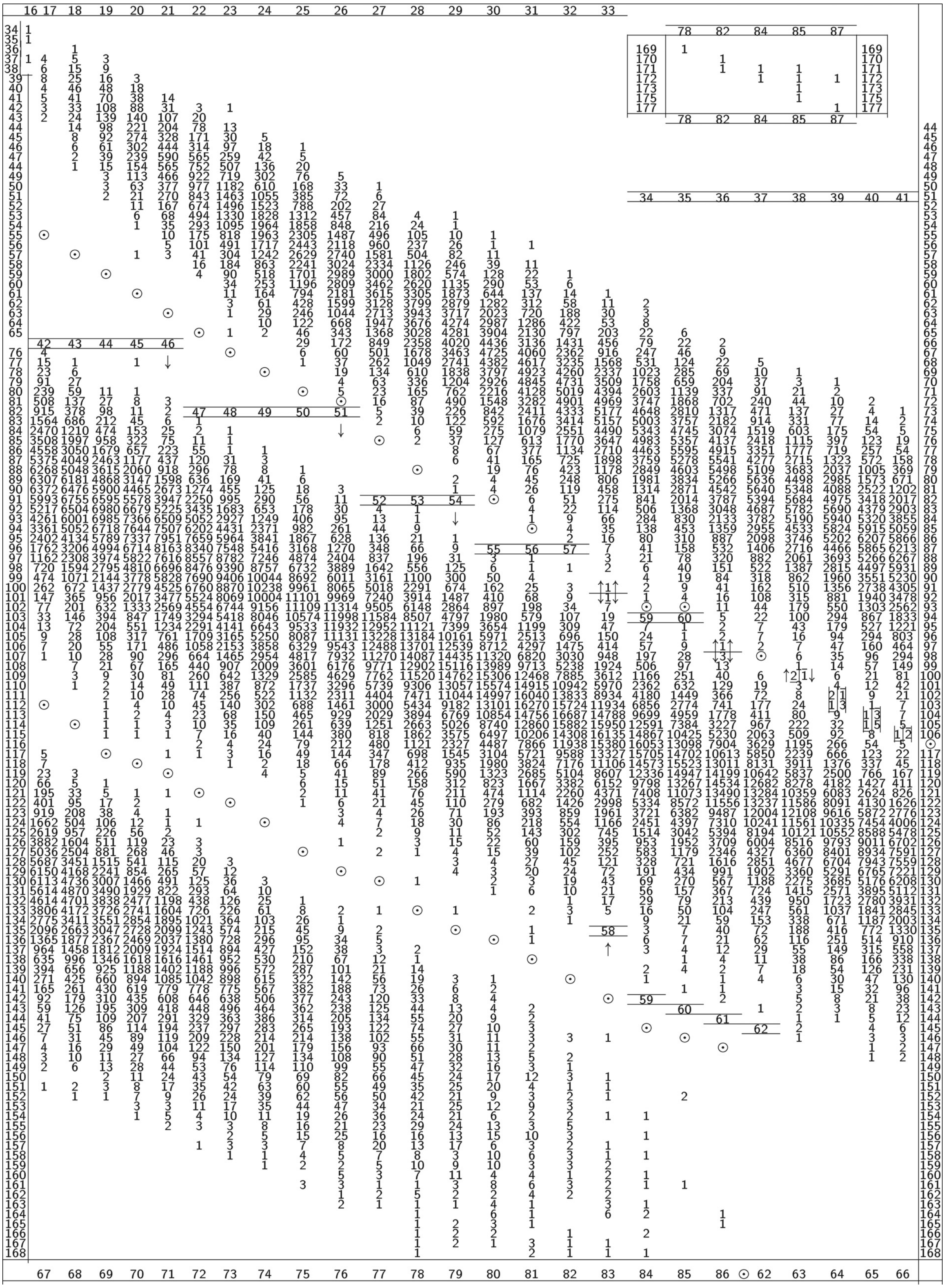


FIG. 12. List of 3 714 503 nonisomorphic 6D KS critical sets from the 236-1216 class; 16 to 87 edges (columns); 34 to 177 vertices (rows); 169-78 to 87-177 sets are shown in the inset.

Critical 8-dim KS sets from the 120-2024 class; 9 to 58 edges (columns) and 34 to 120 vertices (rows)

	9	11	13	15	*16*	17	|	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	
34	2						|			1																			91
36	7	⊗					|																						92
37		1					|																						93
38		7	⊗				|			1	2																		94
39		24	⊗				|				2	2																	95
40		66		⊗			|		1	1	6	3	3		1														96
41		86		⊗			|	1		1	2	6	11	5	1														97
42		128	1				|		1	1	5	13	23	15	5	1													98
43		45	1				|		1		2	11	35	43	20	7		1											99
44		42	7	⊗			|			1	1	15	58	113	96	34	6	2											100
45			10	⊗			|		1	2	1	16	64	214	315	116	29	1											101
46			9				|				1	11	67	299	589	451	160	22											102
47			26				|		1	2	2	6	62	302	958	1244	609	131	11										103
48			31				|		1		1	4	33	251	1180	2482	1954	695	85	4									104
49			21	3			|	1		1	1	4	11	178	1213	3989	4994	2525	539	49	1								105
50			16	2			|					2	7	94	933	4704	9535	7577	2437	291	18								106
51			4	4			|	1			1		6	43	625	4463	14218	17894	9153	1857	157	14							107
52			2	8	1		|					1	2	18	326	3840	16630	32527	25470	7987	1060	77	1						108
53				9									2	8	145	2045	15104	46228	55325	26377	5181	466	24	1					109
54				8				|					3	50	943	10814	50550	93880	69908	21130	2733	157	2					110
55				7		1								2	19	407	5930	42812	124765	141515	65036	12566	1107	58					111
56				3					|					1	6	124	2587	27584	124686	221125	154316	44413	5728	360	14				112
57				4		2								1	1	38	843	13393	93857	258485	278137	121047	22843	1978	78	1			113
58				2		3				|						2	229	4795	223500	58203	372836	246485	686277	8530	508	15	1		114
59						1									1		46	1352	21155	139230	361137	367153	150487	27173	2240	97	3		115
60				⊗		⊗	⊗	⊗	⊗		⊗						8	276	5978	61155	246651	383959	237596	62918	7562	418	10		116
62						1						|					41	8	1276	18859	115068	274469	257663	100166	17092	1361	54		117
63						1	1						|					6	167	3793	35175	127956	180895	103768	25390	2951	145	7	118
64							1							|					13	474	6262	34759	74203	62600	22408	3638	272	11	119
65							2	1							|					24	513	4102	13192	16530	8426	1901	211	14	120
67							1																						|
68							1																						|
69							1	1								|												|
71								1									|											|
72								1										|										72
73								1																					73
77											1																		77
78												2																	78
79											2	1																	79
80									1		2	1	1																80
81										1			4																81
82												1	1																82
83												1	1																83
84													1	2															84
85											⊗			1															85
87													1		1														87
88													1	1	1														88
89															1	1													89
91															1														91
92																	2												92
93																1	1												93
94																			1										94
95																	2												95
97																		1	1										97
98																													98
99																		1	1										99
100																	1												100
	9	11	13	15	*16*	17	*19*	21	*23*	25	*27*	28	29	30	31	34	35	36	37	|	51	52	53	54	55	56	57	58	

FIG. 13. List of 6 925 540 nonisomorphic 8D KS critical sets from the 120-2024 class we obtained on our cluster and of those, denoted as ⊗, obtained by Waegell and Aravind [58] and still not by us.

expressions $r_m e^{in\pi/30}$ (values of constants r_m and n are given in [58]) so that their real and imaginary parts form octads. Using this coordinatization, Waegell and Aravind generate sets of bases (edges) which define their KS sets. We, however, do not need the coordinatization to obtain KS sets. We start with the master set 120-2024 and simply strip off edges. Then, we filter the smaller sets via STATES01 to obtain critical KS sets. We can always add vector components later on, if needed.

The distribution of sets from the 120-2024 KS class is different from the above 6D class as well as from the three of the 4D ones and somewhat similar to the 300-675 4D class with respect to the following feature. The critical sets are split so as to be clustered in two groups of subsets with respect to the number of vertices and edges: first one, sparsely spread, over 9 to about 40 edges and 34 to about 100 vertices and, the second one, densely populated, over about 41 to 58 edges and about 100 to 120 vertices as shown in the table in Fig. 13. The split structure of the 120-2024 class resembles the similarly split structure of the 4D 300-675 class. We conjecture that there is only one or at most a few KS noncritical sets with about 100 vertices and 40 edges which most of the smaller critical sets are subsets of.

Similarly to the 4D classes (with the exception of the 60-105 one) and the 6D class, the number of critical sets which exhibit a parity proof is very small with respect to the total number of critical sets, but on the other hand, parity-proof algorithms used by Waegell and Aravind [58] are very efficient in generating the sets so that the two approaches (via the MMP algorithms for bare hypergraphs and the parity-proof-based ones for vectors corresponding to vertices of hypergraphs) turn out to be complementary. In particular, Waegell and Aravind [58] obtained the following sets which still did not appear in the course of our computer generation so far: 36-11 (Kernaghan-Peres), 38,39-**13**, 40,41,44,45-**15**, 48-**17**,

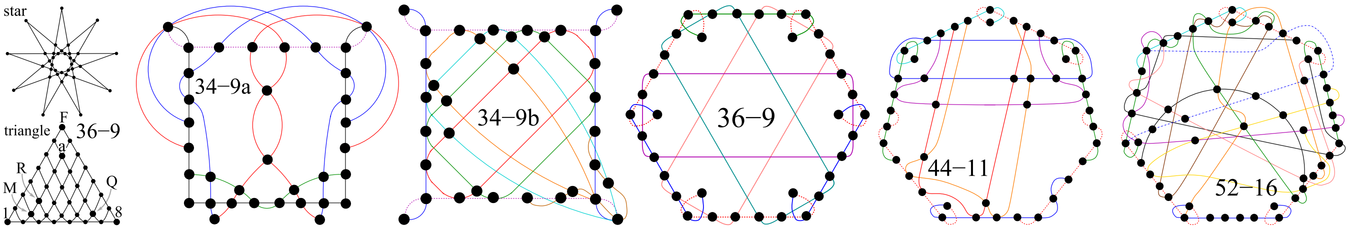


FIG. 14. 8D Kochen-Specker sets: ⋆/△ 36-9 KS set and five chosen critical KS sets from the 120-2024 KS class; see the text for a description of their features.

60-**15,17,19,21,23,27**, and 85-**25** (we do show these sets in the table in Fig. 13 as ⊗); both Waegell and Aravind [58] and we in this paper obtained 34-**9**, 36-**9**, 37-**11**, and 95-**35**; Waegell and Aravind [58] have not obtained all the other sets we obtained in the table in Fig. 13 and most of them they actually cannot obtain due to the features of the parity-based algorithm they make use of but, still, the parity-proof-based programs confirm themselves as a powerful complementary method of providing us with KS criticals since our general MMP hypergraph algorithms are CPU-time demanding.

In Fig. 14, we show five chosen KS criticals from the 120-2024 class. KS criticals 34-9 are the smallest in the class. KS 36-9 is particularly interesting because it can be viewed as an 8D version of 18-9 from Fig. 1(a) with graphically analogous edges where each vertex from the 18-9 is represented by a pair of vertices in the 36-9. KS 44-11 is one of the critical KS sets with the biggest maximal loop (heptagon) among the sets with 11 edges (second smallest number of edges). KS 52-16 has the smallest even number of edges. One of 14 KS 120-58 has the biggest maximal loop, i.e., tetradecagon (14-gon); it is not shown in the figure.

In Sec. V we have seen that the 24-24 class is contained in the 60-105 class and in Sec. VI that the 60-74 class is contained in the 300-675 class. On the other hand, in Sec. VIII we have shown that the ⋆/△ KS set is not contained in the much bigger 236-1216 class of the 6D KS sets. Here, we verified that 8D ⋆/△ 36-9 KS critical set, shown in Fig. 14, is not contained in also much bigger 120-2024 class of 8D KS sets.

The MMP representations of the star and triangle forms (they are mutually isomorphic) of the 36-9 critical are given in the Appendix, Sec. A 5. In Fig. 14, the first three edges correspond to the edges of the triangle as indicated by its vertices `1,8,F` and then the inner vertices are denoted in alphabetical order from left to right from the bottom horizontal ones (indicated by `M` to `Q`) to the single `a` at the top. In contrast to 6D 21-7 set from Fig. 11, this 8D 36-9 can have real vector components from $\{-1,0,1\}$. Coordinatizations for the triangle and for the star are given in the Appendix, Sec. A 5. Interestingly, our program VECTORFIND finds the triangle coordinatization sooner than the one for the star.

The 8D star and triangle set is not smaller than the smallest sets from the 120-2024 KS class as the 6D one is with respect to the smallest sets from the 6D 236-1216 class; the 34-9 sets shown in Fig. 14 are smaller. The 120-2024 class contains at least seven 36-9 criticals but their structure is very different from the star and triangle 36-9 (cf. 36-9 in the middle of Fig. 14).

Via our program SUBGRAPH we prove that the star and triangle 36-9 or any other 36-9 isomorphic to it cannot be obtained by stripping edges and vertices from the master set 120-2024 down to sets with 36 vertices and 9 edges, i.e., that it cannot be a subgraph of the master set and that it therefore does not belong to the 120-2024 class. Of all sets from the 120-2024 class we generated so far, only ca. 0.1% have parity proofs, notably 609 of 6 925 540. The star and triangle 36-9 does have a parity proof, though.

X. 80-265 CLASS OF 16-DIM KS SETS

Harvey and Chryssanthacopoulos constructed an 80-265 KS master set in the 8D real Hilbert space with vector components from the set $\{-1,0,1\}$ [60]. They considered it for four qubits ($2^4=16$) although, theoretically, it can also serve as a KS set for spin-$\frac{15}{2}$ systems. The set has far too many redundant edges, so, Planat promptly designed a procedure to obtain smaller KS sets and he claimed to have obtained three sets with the initial number of vertices: 80-21, 80-22, and 80-23 [57], however, as we show below, his 80-21 and 80-22 are not KS sets and 80-23 is not critical. In this paper, we generate 4.1×10^6 nonisomorphic critical KS sets from the 80-265 master set. We say that KS critical sets that can be generated by stripping the 80-265 master set form the 80-265 class of KS critical sets. The ones we obtained so far are shown in the table in Fig. 15.

The original 80-265 master file printed in [60] took over 11 pages. Its MMP representation is much shorter. Still, it takes over one page. So, we shall consider some smaller examples, but, first, we shall check the sets Planat obtained in [57]. His set 80-21 given by 21 lines of Eq. (17) in [57] has an MMP rendering with 21 edges as given in the Appendix, Sec. A 6. But, this set is not a KS set. For instance, according to our program STATES01 we can assign "1" to `G, H, Y, o, r`, and `u`, so as to exhaust all 21 edges, i.e., when we delete the edges that contain them, then none is left, meaning that each contains one "1" and therefore the set is noncontextual [cf. Fig. 1(a) where, e.g., we can assign "1" to none of vertices `789A` and for which there is always an edge to which one cannot assign "1" at all vertices contained in it]. Then, it is claimed that this 80-21 set together with the 1st line of Eq. (18) from [57], in MMP notation: `2ACEZbhj$(* :<> ?@`, form an 80-22 KS set. However, this 80-22 is not a KS set, either.

All lines from Eqs. (17) and (18), the last line reading `notuwy!#')-:<=>?` in MMP notation, form an 80-23 noncritical KS set. By deleting the first line, `Zbhjprsv$(*-:<=@`, we get a noncritical 80-22 KS set. If we also deleted the eighth line (`HIKLMPQRTUVWbhlm`), we would get a noncritical 80-21 KS set. These sets contain one 80-20 critical set and two nonisomorphic 80-19 criticals, all shown

Critical 16-dim KS sets from the 80-265 class; 11 to 23 edges (columns) and 72 to 80 vertices (rows)

	11	12	13	14	15	16	17	18	19	20	21	22	23	
72	1873	1	14865	12	278									72
73			95559	3618	6583	221	53	2						73
74			221640	20968	60132	6589	2098	73	1					74
75			237116	32967	230599	60783	33983	3899	216					75
76			123769	18717	394131	214168	228704	66601	9206	224				76
77				2303	273552	248801	588721	407809	117876	8322	118			77
78			1	193	89266	99023	490986	742553	446076	70408	2864	29		78
79				9	14360	11250	80457	106160	87482	21627	1592	20		79
80					1448	2392	33573	101333	169197†	88464†	13543	500	7	80
	11	12	13	14	15	16	17	18	19	20	21	22	23	

FIG. 15. List of 4 069 963 nonisomorphic 16D KS critical sets from the 80-265 class. Two 80-19s and one 80-20 KS critical we generated from Planat's [57] noncritical 80-23 are among our 169 197 80-19s and 88 464 80-20s, respectively, indicated by † in the table.

in Appendix A 6. Their maximal loops are pentagons. We obtained them from the aforementioned 80-23,22,21 via our program STATES01.

The goal of [57] was to find small KS sets, but the table in Fig. 15 shows that its non-critical KS set 80-23 is bigger than all 2.5×10^6 KS criticals we generated from the master 80-265 set and listed in the table in Fig. 15. This shows that algorithms for automated exhaustive generation of MMP hypergraphs, although probabilistic until full exhaustion is reached, are indispensable sources for obtaining new KS sets. Still, the 80-20 and 80-19s we obtained with the help of our program STATES01 are not isomorphic to any of the 80-20's and 80-19's we listed in the table in Fig. 15. This is because the probability of generating any specific KS set via our programs MMPSTRIP and STATES01 is very low due to the their probabilistic algorithms. Within established probabilities for obtaining MMP hypergraphs with wanted number of edges and vertices they are generated completely at random.

The 16D KS criticals listed in the table in Fig. 15 have maximal loops in the range from a square to a heptagon as illustrated in Fig. 16. The vector components corresponding to vertices from the set $\{-1,0,1\}$ for the master set listed in [60] can be traced down to any chosen MMP hypergraph from the table in Fig. 15 via any of our programs MMPSTRIP, STATES01, MMPSHUFFLE, etc., or, equivalently, program VECTORFIND can generate the components directly for a given hypergraph.

In contrast to all previous classes of KS sets apart from 4D 60-105, 16D 80-625 class has a significant number of parity proofs, notably, 28%. There are approximately 64% criticals with an odd number of edges but only 44% of them have parity proofs. Also, in contrast to all previous classes, except the 6D 236-1216 class, the KS criticals of the 16D 80-265 class do not exhibit symmetries. They have rather intricate and dense structure. In particular, all vertices share at least two edges and some pairs of edges share eight vertices. Also, in contrast to KS sets from the classes in smaller dimensions (not counting the tentative 24-24 class for which we in Sec. V proved to be contained in the 60-105 class), there are no maximal loops bigger than heptagons. There are approximately 10% of squares, 86% of pentagons, 4.6% of hexagons, and 1% of heptagons.

Why is 77-13 missing, while 76-13 has 123 769 nonisomorphic instances and 78-13 is present, is an open question. The 16D star and triangle set does not admit 0-1 states and is critical and therefore would be a critical KS set if one found a coordinatization for it. We have not found any so far. It has $16+1=17$ edges and $(16+1)16/2=136$ vertices, which are 1.7 times the highest number of vertices of the critical sets from the 80-265 class. Its structure is dissimilar to any obtained set from the 80-265 class so it is very unlikely that it might belong to it, however, for the time being, the program SUBGRAPH which would give us a definitive answer to this question it is still running.

XI. 160-661 CLASS OF 32D KS SETS

Recently, Planat and Saniga, extending Aravind's and DiVincenzo-Peres' generalizations of the Bell-Kochen-Specker theorem [61,62], constructed a 32D KS master set

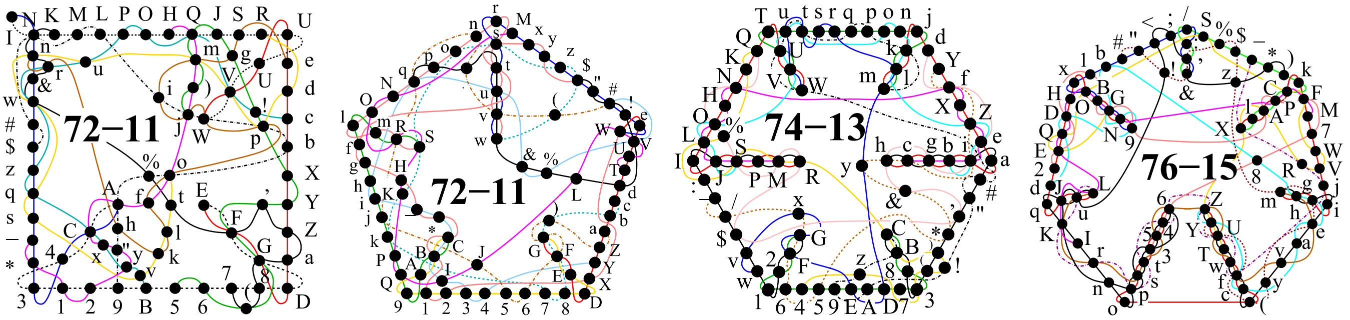


FIG. 16. 16D critical KS sets. The smallest sets with square, pentagon, hexagon, and heptagon maximal loops are shown. MMP hypergraph strings are given in Appendix A 6.

Critical 32-dim KS sets from the 160-661 class; 11 to 29 edges (columns) and 135 to 160 vertices (rows)

	11	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	
135										1									135
⋮										⋮									⋮
144	31	4																	144
145		21		2															145
146		100	15	59	11														146
147		253	54	400	219	124	10												147
148		301	61	1337	1548	1780	553	59		1									148
149		226	26	798	523	675	212	24	1										149
150		217	16	1895	2605	7129	5507	1379	53	1									150
151		127	3	976	513	1350	776	286	13	2	1								151
152		65	5	1647	1733	9609	13324	8862†	1578	60									152
153				852	380	2629	3016	2547	683	80	1		1						153
154			1	739	440	5974	11667	16824	7802	1278	67	1							154
155				283	93	2015	2998	5901	3743	945	91	1							155
156				152	66	2476	5171	14443	13922	6035	1027	66	4			1			156
157				34	8	700	976	4478	6140	4048	974	117	5	1		1			157
158				10	6	451	906	4866	9188	9879	4395	942	82	2	2	2			158
159				3	3	49	43	475	1285	2556	2269	1055	228	17	3	1	2		159
160				1		57	91	861†	2405	4272	4133	2071	562	74	5	2	2	2	160
	11	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	

FIG. 17. List of 254 318 nonisomorphic 32D KS critical sets from the 160-661 class and 52-19 and 60-19 criticals we derived from Planat and Saniga's [63] noncritical 160-21; the latter criticals are included in 8862 52-19s and 861 60-19s, respectively, and indicated by † in the table.

with 160 vertices and vectors and 661 edges with a real coordinatization from the set $\{-1,0,1\}$ [63]. This is a very big set which corresponds to states of five qubits, so, they did not present it in their paper. But, Planat kindly sent us the set in their notation and we translated it to an MMP encoded hypergraph. Planat and Saniga only published a smaller 160-21 KS set they obtained from the master set. MMP hypergraph string of that set is given in Appendix A 7. (Edge 14 in [63] which reads 08 should read 108.)

However, this KS set is not critical and it contains at least two smaller critical KS sets, 160-19 and 152-19 ones. There is no point in giving their MMP representations here because we obtained thousands of smaller KS 32 criticals from the 160-661 master set as shown in the table in Fig. 17. The nonisomorphic KS criticals with the smallest number of edges (11) all have 144 vertices and we show one of them in Fig. 18.

It is interesting that a single KS set with 21 edges (the number of edges of the KS set from [63]) has 9 vertices less than any other set we found (135), indicated by ⋮ in the table in Fig. 17. This might stem from some geometrical structure of the set or its smaller subset. We do not show the 135-21 hypergraph because it has almost twice as many edges as 144-11 and its figure would be much more difficult to read. Also, 135-21 maximal loop is a square and its hypergraph has 47 vertices outside of the loop as opposed to 24 such vertices of the 144-11 hypergraph shown in Fig. 18.

We find the complexity of KS criticals from the 32D 160-661 class similar to the one of the 16D 80-265 class. Not a single vertex shares only one edge, and some pairs of edges share 16 vertices. The distribution of distances between maximal bases was considered in [63] for a single noncritical 160-21 set. In our approach, such a distribution does not play any role for either obtaining thousands of KS criticals or proving that they really are KS sets. The vector components of vertices from the set $\{-1,0,1\}$ for the master set are listed in [63]. As for all the sets from the previous classes given above, we can either trace them or generate them for any given hypergraph.

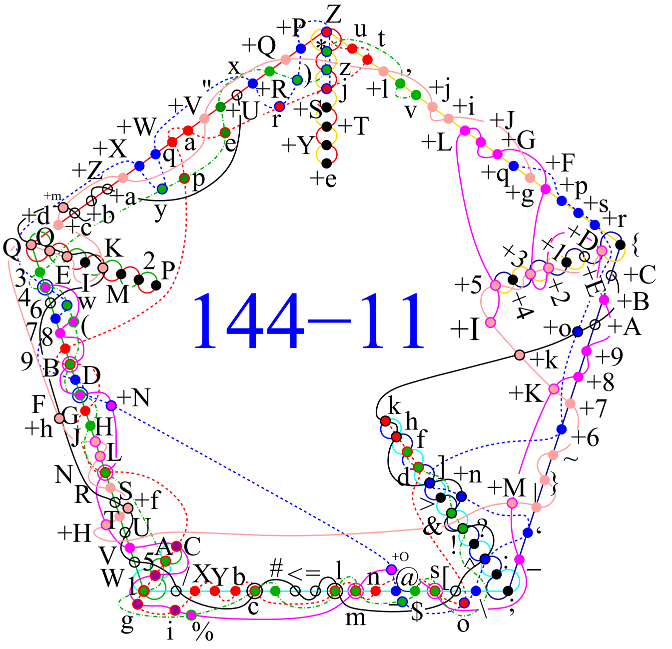


FIG. 18. 144-11; one of 31 smallest 32D critical KS sets.

Of all 2.5×10^5 KS criticals we obtained, only 10.7% have a parity proof. In contrast to all KS sets from the classes in smaller dimensions, there are no maximal loops bigger than hexagons. There are 11.9% of squares, 87.4% of pentagons, and 0.7% of hexagons. The star and triangle KS set, although not admitting 0-1 states and being critical, is far too complicated to be considered here. It has $(32+1)32/2 = 528$ vertices and $32+1 = 33$ edges which makes it far bigger than any of the critical sets from the present class. However, if one found a coordinatization for it, it would be the biggest critical KS set of all known ones.

XII. 3D KS SETS

The successful generations of all the above presented KS sets in up to 32 dimensions were enabled by newly found big master sets and they were in turn derived from various polytopes (like, e.g., 120-cell and 600-cell), or Lie algebras, or some involved individual constructions which made use of geometric symmetries of even-dimensional spaces. Even without the big master sets a direct generation of smaller KS sets is possible via our MMP algorithms [34] because in four- and higher-dimensional space those KS sets are pretty small. Disparately, for the 3D space, we are unaware of a master set and of the few known 3D KS sets, no one is small and all of them are critical and cannot be lessened.

Since it would be very important to find more 3D KS sets to gain a better insight into the structure of contextual KS sets and enable new breakthroughs in their generation and application algorithms and programs, in this section we give MMP representations and KS hypergraphs of the four known (the only known ones) 3D KS sets and one that was claimed to be of such kind (the Yu-Oh 13-set), but is not, as we show below.

The full specification of all vertices (their vector components) is, as shown by Larsson [64] and Pavičić, Merlet, McKay, and Megill [34], indispensable "for an experimental realization, which involves procedures equivalent to basis rotations" ([65], p. 332, end of the 1st par.). For example, spin-1 particle flying through a sequence of generalized Stern-Gerlach devices whose filters or paths correspond to three orthogonal *eigenprojections* of the spin observable [33] and we would not have a correct measurement statistics if we ignored some of the vertices present in particular edges.

As shown in Fig. 19, Bub's [46], Conway and Kochen's [49], Peres' [43], original Kochen and Specker's [30] KS sets and Yu and Oh's non-KS set [66], have 49, 51, 57, 192, and 25 vertices, respectively (and 36, 37, 40, 118, and 16 edges, respectively). In Fig. 19, the vertices that share only one edge are denoted by fully grayed dots and gray ASCII characters. If we ignored them in an implementation, we would be left with 33, 31, 33, 117, and 13 vertices, but then the measurements would give us incorrect data as we explained above. Surprisingly, in all presentations of their KS sets, the aforementioned authors simply dropped the (gray) vertices that shared only one edge in an attempt to present their KS sets as being smaller and therefore more attractive for possible implementations.

Yet, all those vertices or vectors have definite vector components in the coordinatization they made use of. Thus, it is just the visual presentation of these KS sets in the original papers and subsequent reviews in numerous articles and books of these sets that are misleading, not the actual structures of them (which are perfectly correct).

Yu and Oh published a paper [66] in which they introduced a set with 13 vertices which they call a 13-ray set: 13-vertex set in our notation. The set is displayed in Fig. 19 where the 13 vertices are shown as red dots (in online version; black dots in printed version). Yu and Oh dropped the vertices that share only one edge, shown as gray dots in the figure (12 of them), following the aforesaid manner. In our figure, we see that after restoring the dropped vertices it is possible to assign 0's and 1's to vertices from all edges. So, the Kochen-Specker Theorem 1 tells us that Yu-Oh's 13-vertex set is not a KS set. Actually, Yu and Oh themselves cite the Bell-Kochen-Specker theorem in the same wording as in Theorem 1 and admit that their set satisfies the conditions of the theorem ([66], p. 3, top). That can be formulated as the following lemma.

Lemma 5. Yu-Oh's 13-vertex set is not a KS set.

Proof. It is possible to assign 0's and 1's to vertices in such a way that no two orthogonal directions are both assigned 1 and no three mutually orthogonal directions are all assigned 0 as shown by encircled 1's in Fig. 19. ■

Yu and Oh admit the validity of Lemma 5 as follows: "The KS value assignments to the 13-ray set are possible; i.e., no logical contradiction can be extracted by considering conditions 1 and 2 (of Theorem 1) only." Yet, they claim to have "proved the original KS theorem" [66]. However, Lemma 5 appears to prove the contrary. This may be the result of misapplied terminology. In the paper they proceed to define a new kind of contextuality through their inequalities (2), (3), and (4) applied to their 13-non-KS-set, and then they

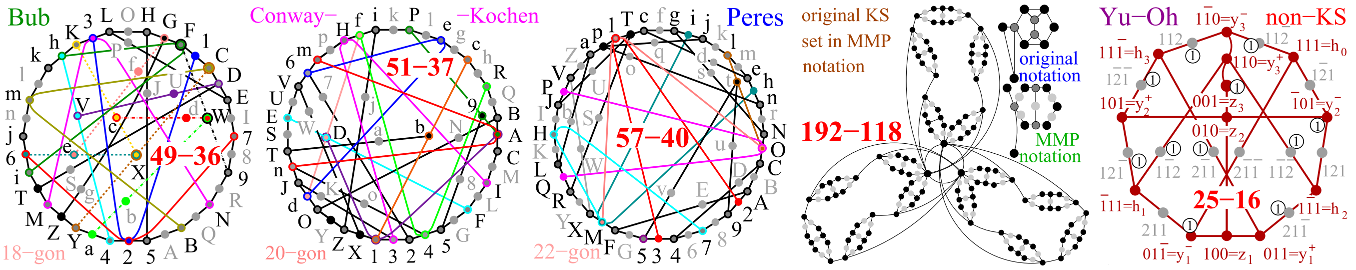


FIG. 19. Four 3D KS sets: Bub's 49-36, Conway-Kochen's 51-37, Peres' 57-40, and original Kochen-Specker's 192-118, and Yu-Oh's non-KS set named "13-vertices (-rays) set" according to 13 red (online; black in print) vertices; the components of each vector (vertex, ray) are from the set {0, ± 1, ± 2}; $\bar{1}$ and $\bar{2}$ stand for −1 and −2, respectively.

mistakenly claim that their proof of such a newly defined contextuality amounts to the proof of the Kochen-Specker theorem. Only a KS set can be a proof of the KS theorem since a violation of conditions 1 and 2 of the theorem is tantamount to a definition of a KS set and therefore no non-KS can prove the theorem since it does not violate them [67]. This does not mean that the contextuality Yu and Oh proved for their 13-set is wrong. This only means that via such a contextuality for their 13-set one cannot prove the Kochen-Specker theorem simply because the set is not a KS set.

Hence, we are left with the four big KS sets as the only known 3D KS sets. (Gould and Aravind have proven that the so-called Penrose's 3D KS set is isomorphic to Peres' one [68].) It is well known that all of them are critical and our program STATES01 confirms that. So, we cannot use them to generate smaller KS sets. (By the way, Yu-Oh's 13-vertex set is a subset of Peres' critical set and its criticality is yet another avenue of proving that Yu-Oh's set cannot be a KS set and therefore that it cannot prove the KS theorem [67].) But when we look at their MMP hypergraphs we notice that the number of gray dots, i.e., the number of vertices that share only one edge, increases with the total number of vertices within a KS set, in contrast to the opposite trend of KS sets from the 4D 60-105 class; cf. red circles in 29-16, 30-16, 31-18, and 60-40 in Fig. 6. In particular, there are 16, 20, 24, and 75 such vertices (gray dots) in Bub, Conway-Kochen, Peres, and Kochen-Specker's hypergraphs in Fig. 19, respectively.

Therefore, we conjecture that more complex noncritical KS sets, interwoven similarly to higher-dimensional KS sets, with comparatively low number of vertices, approximately 50, might be found on clusters and supercomputers and used to generate smaller 3D KS criticals. This is a work in progress.

XIII. DISCUSSION

In the past 10 years, the exploration and generation of contextual sets, in particular, Kochen-Specker (KS) sets, received a lot of attention (see Sec. I) both for their possible applications and implementations and for their further theoretical usage and development in quantum mechanics and quantum information. The approaches to generation of KS sets diversified and many partial results were achieved, recently. Therefore, we have focused our efforts on the unification of results, features, structure, and mutual relations of different KS sets as well as on the development of technique and method of their arbitrary exhaustive generation and handling.

In pursuing this goal, we have concentrated neither on immediate experimental implementation (small sets) nor on the standard parity-proofs-based algorithms. Instead, we have made use of the general MMP hypergraph language by means of which we generated a large number of additional types of contextual KS critical sets and numerous nonisomorphic instances within each of them, which mostly cannot be generated by other known algorithms. The approach also gave us the results we were not originally concerned with, such as an abundance of small KS sets, and in addition provided us with an explanation why the other approaches, like parity-proof ones, failed to spot them; it turns out that only a very few KS sets have a parity proof (in some classes under 1% of sets and in some none at all) what makes them completely invisible for the parity-proof-based algorithms and programs, predominant in the literature.

Instead of parity proofs of only some KS sets with a particular structure, the MMP hypergraph algorithms and methods enable direct numerical proofs of the KS theorem for any chosen KS set via literal verifying of KS theorem conditions: program STATES01 gives a maximal number of 1's for a chosen KS set and after deleting all edges that contain these 1's at least one edge should remain. This is all automated, but the user can easily check the output MMP hypergraph strings by hand. When the MMP hypergraphs are drawn as figures, the proofs also become "visual" as indicated in Figs. 1(a) and 3 (26-13) by dashed red ellipses.

We developed the hypergraph approach to KS sets introduced in Sec. II from the lattice theory of Hilbert spaces and the way of assigning of 0-1 states to vertices of hypergraphs we took over from the methods of dealing with discrete states defined on those lattices. In particular, we redesigned our programs for analyzing the Hilbert lattice features and turned them into the programs we used to build up an MMP-hypergraph-based language (specified in Sec. II) for generating, analyzing, filtering, and modifying KS sets. We made use of this language to obtain numerous KS sets and classes and their features in 4D (in Secs. IV, V, VI, and VII), 6D (in Sec. VIII), 8D (in Sec. IX), 16D (in Sec. X), and 32D (in Sec. XI) Hilbert spaces. We also reviewed the 3D KS sets in Sec. XII. In the table in Fig. 20 we list the most important properties of critical KS sets we obtained and compare them with ones obtained previously.

The large variety of KS sets we obtain provide a much greater choice for KS experiments and insight into their properties and the properties of gates that handle them as well as the properties of quantum sets in general. While smaller KS sets are currently preferred for a feasibility of experimental implementations, in the future other sets that are intrinsically different (i.e., nonisomorphic) may become desirable for more sophisticated experiments, verification of the KS theorem with different setups, etc., especially because bigger sets do not require higher efficiency of measurements but only a higher number of measurements. Since the sets we found are critical, there will be no redundancy in any experimental setup making use of them.

Finally, we want to stress that our generation of KS sets in 16D and 32D spaces allowing for their implementation by means of four and five qubits, respectively, is (as, actually, all generations in this paper) vector or vertex based and therefore complementary to a recent operator-based generation of KS sets for four, five, and six qubits (explicitly, and more of them, in principle) by Waegell and Aravind [69]. Building a correspondence between the two approaches (via eigenvectors of their operators) is a work in progress so that we, as of yet, cannot say to which extent our results overlap. We can only say that we have not obtained KS criticals with nine edges (bases, in their terminology) for four and five qubits, as they have. Our minimal number of edges for the corresponding two classes is 11, as shown in the tables in Figs. 15, 17, and 20. We did not include 16D and 32D KS criticals with nine edges from [69] into our tables because we were not able to find out whether they, respectively, belong to the 80-265 and 160-661 classes or not.

Properties of critical non-isomorphic KS sets from considered KS classes												
Dimension	3-dim	4-dim					6-dim		8-dim		16-dim	32-dim
Class	4 sets	24-24	60-74	60-105	300-675	148-265	☆/△	236-1216	☆/△	120-2024	80-265	160-661
Space, vectors	r(eal)	r	r	c(omplex)	r	c	c	r	r	c	r	r
Contained in	na	60-105	300-675	no	na	no	no	na	no	na	na	na
№ of qubits	na	2	na	2	na (2?)	na	na	na	na	3	4	5
Parity (fraction)	0	1	7.8×10^{-5}	0.76	0.14^*	0	1	2.2×10^{-6}	1	10^{-4}	0.28	0.11
Min № of edges	36	9	13	9	13	23	7	16	9	9	11	11
Max № of edges	118	15	41	40	188	97	7	87	9	58	23	29
Min № of vertices	49	18	26	18	26	40	21	34	36	34	72	135
Max № of vertices	192	24	60	60	283	147	21	177	36	120	80	160
Edges share ≥ 2 v	−	+	−	+	−	−	−	+	−	+	+	+
Max loops, n-gons	18···22	6	8···18	7···17	8···57	10···36	3	4···14	3	4···14	4,5,6,7	4,5,6
New types(%)	0	0	18.3	67.5	57	100	0	100	100	94.6	99	99
New sets (ca. №)	0	0	10^9	7×10^6	10^3	2.5×10^5	0	3.7×10^6	1	6.2×10^6	2.5×10^6	2.5×10^5
References	[45, 48] [29, 42]	[44, 46] [31, 64]	[49, 50] [27, 37]	[8] [51]	[52]	[54]	[24]	[55]	./.	[43, 58] [57, 59]	[60] [57]	[63]

FIG. 20. List of properties of generated critical KS sets and their comparison with the previously obtained sets; "na" stands for "not applicable," e.g., for the biggest known classes; "2?" refers to the claim in [53] that the operators defining the 300-675 might be redefined to allow a representation via 2 qubits; * in 0.14* for 300-675 indicates unevenly distributed parity proofs: all lower criticals have it while all higher ones (211-127 to 283-188) lack it; "edges share ⩾2 v" = edges share 2 or more vertices, i.e., intersect each other two or more times; "new types" "0" for, e.g., 24-24 means that there are no new types of 24-24 KS criticals in this paper (they are only elaborated on and discussed here); "18.3" (for 60-74) means: 28 (new types obtained in this paper; others were obtained in [27,38,50,51]) / 153 (total number of types) = 18.3%; "100"(for 236-1216) means that all types are from this paper; "new sets" give number of sets obtained in this paper.

The generation of KS critical sets we presented in this work is, with our algorithms and programs, straightforward but demanding and CPU-time consuming. The jobs require cluster and grid calculations and even with them it might take months to obtain a required or desired particular set which might be needed in elaboration, confirmation, or checking particular assumptions about construction, unification, geometry, computation, or implementation of contextual sets. We provide samples of the KS criticals in the MMP hypergraph notation in the Supplemental Material [28]. They are just a tiny fraction of TBs of data we generated but the reader can obtain any KS sets from us upon "appropriate" request (e.g., the 60-74 file has 231 GB). Also, the reader may generate any KS set by making use of our programs that are freely available at our repository http://goo.gl/xbx8U2.

ACKNOWLEDGMENTS

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APPENDIX: MMP HYPERGRAPH STRINGS REFERRED TO IN THE ARTICLE

1. Section IV

Master set 60-75: 1234, 5678, 9ABC, DEFG, HIJK, LMNO, PQRS, TUVW, XYZa, bcde, fghi, jklm, maSK, lZRJ, kYQI, jXPH, ieWO, hdVN, gcUM, fbTL, nopq, rstu, vwxy, yuqG, xtpF, wsoE, vrnD, pdYC, qeZB, ocXA, nba9, uROC, tQNB, sPM9, rSLA, wUHC, xVIA, yWJ9, vTKB, jgEB, liGA, khF9, mfDC, pLJ8, qMI7, nNH6, oOK5, vhX8, ygY5, xfZ6, wia7, ukc6, rjd7, tlb5, sme8, UQG8, TRF7, VPD5, WSE6, tXW4, rZU3, sYT2, uaV1, ofQ1, piP3, ngR4, qhS2, xjO2, wkL4, vlM1, ymN3, eHF1, cJD2, bIE3, dKG4.

Master set 60-74: 1234, 5678, 9ABC, DEFG, HIJK, LMNO, PQRS, TUVW, XYZa, WOGC, VNFB, UMEA, TLD9, aSK8, ZRJ7, YQI6, XPH5, bcde, fghi, jklm, mie4, lhd3, kgc2, jfb1, ndIB, oeJA, pcK9, qbHC, reF8, sdE5, tbD6, ucG7, rfY9, uhXA, tiaB, sgZC, nkT8, plV6, qmU7, ojW5, piRE, ogSD, qhQF, nfPG, smNK, ujLI, rlMH, tkOJ, sTQ1, uVS4, tUP3, rWR2, oYN3, nZM4, qaL2, pXO1, qpon, utsr, vePL, wcQM, xbRN, ydSO, vkYE, yjZF, wmXD, xlaG, wfVJ, xgUI, yiTH, vhWK, x954, vB71, yA62, wC83.

60-39: 4123, 3hsS, SQRP, PcwO, OLMN, NiX5, 5786, 6teF, FDEG, GVgp, pnqo, oabA, ABC9, 9fxK, KHIJ, JUd4,,, bcde, ujHZ, 8vTl, 2mMn, 7IRq, EyY1, WLrB, QCkD, IlXt, Uxj7, buE5, n8QH, 1KcT, arKk, hnUE, m5Ww, t1gQ, yHiA, 7Dsc, Of8Y, VbI2, Z2Dv, AtOU.

60-41: 2341, 1gQt, t6eF, FjBS, SydM, MLON, NiX5, 5mWw, whCI, I7Rq, qnpo, oAba, arkK, Kfx9, 9EPl, l8Tv, vZD2,,, 5678, 9ABC, DEFG, HIJK, PQRS, bcde, VgpG, hs3S, 2mMn, EyY1, WLrB, JUd4, QCkD, Uxj7, buE5, n8QH, 1KcT, 9Zdq, 7Dsc, CTuM, Of8Y, VbI2, iTFq, BcnX.

2. Section V

Master set 60-105: 1234, 5678, 9ABC, DEFG, HIJK, LMNO, PQRS, TUVW, XYZa, bcde, fghi, jklm, nopq, rstu, vwxy, 12FG, 12RS, 13MO, 13UW, 14gh, 14pq, 23fi, 23no, 24LN, 24TV, 34DE, 34PQ, 56JK, 56VW, 57EG, 57Ya, 58kl, 58oq, 67jm, 67np, 68DF, 68XZ, 78HI, 78TU, 9ANO, 9AZa, 9BIK, 9BQS, 9Ccd, 9Cnq, ABbe, ABop, ACHJ, ACPR, BCLM, BCXY, DERS, DFYa, DGde, DGtu, EFbc, EFrs, EGXZ, FGPQ, HIVW, HJQS, HKhi, HKsu, IJfg, IJrt, IKPR, JKTU, LMZa, LNUW, LOlm, LOru, MNjk, MNst, MOTV, NOXY, PSkm, PSxy, QRjl, QRvw, TWce, TWwy, UVbd, UVvx, Xagi, Xavy, YZfh, YZwx, bctu, bdwy, benq, cdop, cevx, ders, fgsu, fhvy, fipq, ghno, giwx, hirt, jkru, jlxy, jmoq, klnp, kmvw, lmst.

Master set 24-24 as a subgraph of the master set 60-15 with original complex vector components: 1234, 5678, 9ABC, DEAC, FG9B, 9CHI, ABJK, HJKI, DFGE, 58JI, 67HK, 56GE, 78DF, LMNO, N4BC, O39A, 1MKI, 2LHJ, 1LFE, 2MDG, 57O4, 68N3, NO34, 12LM.{1=(1,0,0,0), 2=(0,1,0,0), L=(0,0,1,0), M=(0,0,0,1), 5=(1,1,*i*,*i*), 6=(1,-1,*i*,-*i*), 7=(1,1,-*i*,-*i*), 8=(1,-1,-*i*,*i*), N=(1,1,0,0), O=(1,-1,0,0), 3=(0,0,1,1), 4=(0,0,1,-1), D=(1,0,*i*,0), F=(0,1,0,*i*), G=(1,0,-*i*,0), E=(0,1,0,-*i*), 9=(1,1,*i*,-*i*), A=(1,1,-*i*,*i*), B=(1,-1,*i*,*i*), C=(1,-1,-*i*,-*i*), H=(1,0,0,*i*), J=(1,0,0,-*i*), K=(0,1,*i*,0), *I*=(0,1,-*i*,0)}

20-11a with coordinatization from 60-105: 1234, 5678, 19A8, 5BC4, DEFG, HIJK, 6BDF, A7HJ, C3HI, 29DE, HKDG.{1=(1,0,0,0), 2=(0,1,0,0), 9=(0,0,1,0), 5=(1,1,*i*,*i*), 6=(1,-1,*i*,-*i*), B=(1,1,-*i*,-*i*), C=(1,-1,0,0), 3=(0,0,1,1), 4=(0,0,1,-1), A=(0,1,0,*i*), 7=(1,0,-*i*,0), 8=(0,1,0,-*i*), H=(1,1,*i*,-*i*), *I*=(1,1,-*i*,*i*), J=(1,-1,*i*,*i*), K=(1,-1,-*i*,-*i*), D=(1,0,0,*i*), E=(1,0,0,-*i*), F=(0,1,*i*,0), G=(0,1,-*i*,0)}

18-9 with coordinatization from 60-105: 1234, 1567, 869A, BACD, ECFG, H3IF, H4B9, 25ED, 87IG.{1=(1,0,0,0), 2=(0,1,0,0), 5=(0,0,1,0), H=(1,-1,0,0), 3=(0,0,1,1), 4=(0,0,1,-1), 8=(1,0,-1,0), 6=(0,1,0,-1), 7=(0,1,0,1), B=(1,1,-1,-1), 9=(1,1,1,1), A=(1,-1,1,-1), E=(1,0,0,-1), C=(0,1,1,0), D=(1,0,0,1), I=(1,1,1,-1), F=(1,1,-1,1), G=(1,-1,1,1)}

20-11b with coordinatization from 60-105: 1234, 1567, 2589, ABCD, EFGH, 8IFH, ADI9, J7EH, J6CD, K4GH, K3BD.{1=(1,0,0,0), 2=(0,1,0,0), 5=(0,0,1,0), K=(1,1,0,0), 3=(0,0,1,1), 4=(0,0,1,-1), J=(1,0,-1,0), 6=(0,1,0,-1), 7=(0,1,0,1), A=(1,1,-1,-1), B=(1,-1,-1,1), C=(1,1,1,1), D=(1,-1,1,-1), 8=(1,0,0,-1), I=(0,1,1,0), 9=(1,0,0,1), E=(1,1,1,-1), F=(1,1,-1,1), G=(1,-1,-1,-1), H=(1,-1,1,1)}

3. Section VI

38-19: 4123,3C6L,LVYE,EDGF,FOQP,PZaS,SNRJ,JIKH,HTXM,M794,,,5678,9ABC,B5N1,8OA2,TUVW,baXG,ZYcI, QKbU,cRDW

42-21: 2143,36CP,PIOe,eGcW,WVXU,ULdg,gFHa,aRZY,YbfT,TJNS,S8A2,,,5678,9ABC,DEFG,HIJK,LMNO,479Q, 15BR,Qbcd,EVMZ,DKfX.

48-25: 1243,36CP,Pcde,eVUX,XQYZ,ZmiE,EFGD,DabO,OMNL,LIhk,kjgR,R5B1,,,5678,9ABC,HIJK,479Q,28AS, SJTU,RVNW,GHfg,Sahi,Fjdl, cTYW,SfMl,RKmb.

221-127: ++S++T++U++V, ++V++5++6++7, ++7+j+k+’’, +’’+z+!+‘, +‘+^+_++R, ++R++O++P++Q, ++Q+g+i+h, +h+V+e+>, +>+<+=+?, +?|}+T, +T+R+m+S, +SQ#++E, ++E++C++D++F, ++Fy+fz, zRt++8, ++8++9++B++A, ++A+b+c+a, +a{+/J, JN>+M, +M+L++3+|, +|+{+~+}, +}+&+’+(, +(D+d++L, ++L++K++M++N, ++NU;S, S!+s@, @f+Ed, dcu++H, ++HBI+$, +$+%++4+#, +#AL+B, +B+9+A+[, +[+@+\+], +]+w+y+x, +x6^E, ECs+o, +o+p+q+r, +r+I+J+K, +K+G++J+H, +H-+:+8, +8VW&, &$+W%, %O+P++G, ++GXYZ, Zn+3h, hgxi, iP+t++S,,,++G++H++I++J, ++1++2++3++4, +-+/+:+;, +)+*+;+~, +s+t+u+v, +l+m+n++U, +d+e+f++B, +X+Y+Z++A, +W+n+!++A, +V+Z+z++6, +U+i+\++6, +N+O+P+Q, +F+c++6++I, +C+D+E+J, +6+7+8+[, +5+W++2++6, +3+4+(++A, ~+1+2++I, _‘{+v,]^+*++1, @[\+Y, =>?+k, <\+2++5, /:;++P, ’()*, #-+u++O, ‘‘*+K+>, !+J+Q++R, wx+7++F, vx+U+>, stu<, ru+4+q, pq?++5, o)+‘++J, mn++9++T, l*+y+_, jk[+t, ef(+b, ab@+{, TU+g++H, Pv+M+^, NR+I++M, Mq++M++O, KL-++7, w+P+=++U, &:+U+%, HI’+x, GNry, EFQ(, M/+4+E, QT+a+’, Tf+l++D, Su+D++Q, CD+J++8, F+f+[+_, t+3+B++N, s+K++F++K, u+#+>++C, 9+3+J+f, 8k++F++V, 7+p++8++K, 6=+@++B, ^+O+d+j, 5H++P++V, IKh++R, 79O++C, H{+A++B, 5n=+X, JZ+<++S, MS]++D, 9L++9++L, $^+[++E, G+o+(++C, k+3+T+^, +I+V+(++S, Or+K+d, 4=+w++E, 3G+e++K, 2:+X++5, 1Cw+^.

4. Section VII

40-23: `3124, 49AB, BaVL, LMNE, ECD8, 8567, 7IJK, KRSQ, QOPH, HFG3,,, TUV6, WXP5, YZSD, abXG, cZN2, deR1, ebMJ, dWCA, cUIF, cbOC, daYI, WSLF, eZVP.` 1=$(1,0,\omega^2,\omega^2)$, 2=$(1,\omega,0,-\omega^2)$, 3=$(1,-\omega,-\omega^2,0)$, 4=$(0,1,-\omega,\omega)$, 5=$(1,\omega,0,-\omega)$, 6=$(1,0,\omega,\omega)$, 7=$(0,1,-1,1)$, 8=$(1,-\omega,-\omega,0)$, A=$(1,0,0,0)$, B=$(0,1,-\omega^2,1)$, C=$(0,0,0,1)$, D=$(1,-1,-\omega^2,0)$, E=$(1,-\omega^2,-1,0)$, F=$(1,\omega,0,-1)$, G=$(0,1,-\omega,\omega^2)$, H=$(1,0,\omega^2,1)$, I=$(1,0,1,1)$, J=$(1,1,0,-1)$, K=$(1,-1,-1,0)$, L=$(1,\omega^2,0,-\omega^2)$, M=$(0,1,-\omega,1)$, N=$(1,0,1,\omega^2)$, O=$(1,-\omega^2,-\omega^2,0)$, P=$(1,\omega^2,0,-1)$, Q=$(0,1,-1,\omega)$, R=$(1,0,1,\omega)$, S=$(1,1,0,-\omega)$, U=$(0,1,-\omega^2,\omega^2)$, V=$(1,-\omega^2,-\omega,0)$, W=$(0,0,1,0)$, X=$(1,1,0,-\omega^2)$, Y=$(1,0,\omega^2,\omega)$, Z=$(0,1,-\omega^2,\omega)$, a=$(1,0,\omega,\omega^2)$, b=$(1,-1,-\omega,0)$, c=$(1,-\omega,-1,0)$, d=$(0,1,0,0)$, e=$(1,0,\omega,1)$

49-27: `3241, 1675, 5XHW, WCLn, nEmj, jiIc, caZb, blJD, DghO, OPQN, NRS9, 98A3,,, 3BCD, 3EFG, 4HIJ, 4KLM, TUVS, 5YFZ, aUGM, dOEH, deCZ, dRfM, geXK, 6kfh, 6UCI, lkFL, 7RBm.` 1=$(0,0,0,1)$, 3=$(1,0,0,0)$, 4=$(0,0,1,0)$, 5=$(1,1,-\omega,0)$, 6=$(1,\omega^2,-\omega^2,0)$, 7=$(1,\omega,-1,0)$, 9=$(0,1,-1,-\omega^2)$, B=$(0,1,\omega^2,\omega)$, C=$(0,1,1,1)$, D=$(0,1,\omega,\omega^2)$, E=$(0,1,1,\omega)$, F=$(0,1,\omega,1)$, G=$(0,1,\omega^2,\omega^2)$, H=$(1,-1,0,\omega)$, I=$(1,-\omega^2,0,\omega^2)$, J=$(1,-\omega,0,1)$, K=$(1,-\omega^2,0,1)$, L=$(1,-\omega,0,\omega)$, M=$(1,-1,0,\omega^2)$, N=$(1,-1,-1,0)$, O=$(1,0,1,-\omega)$, R=$(1,0,1,-\omega^2)$, S=$(1,1,0,\omega^2)$, U=$(1,0,\omega^2,-\omega^2)$, W=$(1,0,\omega,-\omega)$, X=$(0,1,\omega,\omega)$, Z=$(1,-1,0,1)$, a=$(1,1,-\omega^2,0)$, b=$(1,0,\omega^2,-1)$, c=$(0,1,\omega^2,1)$, d=$(1,1,-1,0)$, e=$(1,0,1,-1)$, f=$(0,1,1,\omega^2)$, g=$(1,\omega^2,-1,0)$, h=$(1,-\omega^2,0,\omega)$, j=$(1,0,\omega,-\omega^2)$, k=$(1,0,\omega^2,-\omega)$, l=$(1,\omega,-\omega^2,0)$, m=$(1,-\omega,0,\omega^2)$, n=$(1,\omega,-\omega,0)$

5. Section IX

8D ★: `12345678,89ABCDEF,FGHI4JKL,L7MNBOPQ,QERSI3TU,UK6VNAWX,XPDYSH2Z,ZTJ5VM9a,aWOCYRG1.`

8D Δ: `12345678, 89ABCDEF,FGHIJKL1,L2MRVYaE,KM3NSWZD,JRN4OTXC,IVSO5PUB,HYWTP6QA,GaZXUQ79.,`

Coordinatization for Δ: 1=(0,0,0,0,0,0,0,1), 2=(0,0,0,0,0,0,1,0), 3=(0,0,0,0,0,1,0,0), 4=(0,0,0,0,1,0,0,0), 5=(0,0,1,1,0,0,0,0), 6=(0,0,1,-1,0,0,0,0), 7=(1,1,0,0,0,0,0,0), 8=(1,-1,0,0,0,0,0,0), 9=(0,0,0,0,0,0,1,1), A=(0,0,1,1,1,-1,0,0), B=(1,1,0,0,0,0,-1,1), C=(1,1,0,0,0,0,1,-1), D=(0,0,1,0,-1,0,0,0), E=(0,0,0,1,0,1,0,0), F=(0,0,1,-1,1,1,0,0), G=(0,0,0,1,1, 0,0,0), H=(0,0,1,1,-1,1,0,0), I=(1,0,0,0,0,0,1,0), J=(0,0,1,0,0,-1,0,0), K=(1,0,0,0,0,0,-1,0), L=(0,1,0,0,0,0,0,0), M=(0,0,1,0,1,0,0,0), N=(0,0,0,1,0,0,0,0), O=(1,-1,0,0,0,0,-1,-1), P=(0,0,0,0,1,1,0,0), Q=(1,-1,0,0,0,0,-1,1), R=(1,0,0,0,0,0,0,1), S=(0,1,0,0,0,0,0,-1), T=(0,1,0,0,0,0,-1,0), U=(0,0,1,-1,1,-1,0,0), V=(0,0,1,-1,-1,1,0,0), W=(1,1,0,0,0,0,1,1), X=(0,0,1,0,0,1,0,0), Y=(1,0,0,0,0,0,0,-1), Z=(1,-1,0,0,0,0,1,-1), a=(0,0,1,1,-1,-1,0,0)

Coordinatization for ★: 1-8—as for △, 9=(1,1,0,0,0,0,-1,1), A=(0,0,1,1,1,-1,0,0), B=(0,0,0,0,0,0,1,1), C=(0,0,1,-1,1,1,0,0), D=(0,0,0,1,0,1,0,0), E=(0,0,1,0,-1,0,0,0), F=(1,1,0,0,0,0,1,-1), G=(0,0,1,0,0,-1,0,0), H=(1,0,0,0,0,0,0,1), I=(0,0,0,1,0,0,0,0), J=(1,-1,0,0,0,0,-1,-1), K=(0,1,0,0,0,0,-1,0), L=(0,0,1,0,0,1,0,0), M=(0,0,1,-1,1,-1,0,0), N=(1,-1,0,0,0,0,-1,1), O=(0,0,0,1,1,0,0,0), P=(0,0,1,1,-1,-1,0,0), Q=(1,-1,0,0,0,0,1,-1), R=(1,0,0,0,0,0,-1,0), S=(0,0,1,0,1,0,0,0), T=(0,1,0,0,0,0,0,-1), U=(1,1,0,0,0,0,1,1), V=(0,0,0,0,1,1,0,0), W=(0,0,1,1,-1,1,0,0), X=(1,0,0,0,0,0,0,-1), Y=(0,1,0,0,0,0,0,0), Z=(0,0,1,-1,-1,1,0,0), a=(1,0,0,0,0,0,1,0).

6. Section X

16D 80-21: `Zbhjprsv$(*-:<=@, HIPQZbdehjlmpqvx, Zehmoqwxz!’’#’)>?, 12457BCGLMPQZbdm, 378GIJKLOPUVXgik, 12346789ACEGJKOV, 16ACNOVWXYakny!#, HIKLMPQRTUVWbhlm, KRVWXYabcfghiklm, 129AKRVWbhlmqrv’’, 37BFXYfgstvwxy’’#, 123456789ABCDEFG, 23ABCDEG$%:;<>?@, Xafiuwy!$()/:;=?, 46CEpsuwxyz!$(/;, ILQTnot#%&’*<=?@, 5BDFLMNRSTUWrsz’’, 5BDFHINPQRSWpqvx, XYacfgik$%&(*/;@, HIKLMOPQSTUW&(;@, 16ACXYadeklm%&/;.`

16D 80-20: `123456789ABCDEFG, HIJKLMNOPQRSTUFG, VWXYZabcdeRSTUDE, fghijklmndeQABCG, opqrstmn456789BC, uvwxyz!’’jklbce3E, #$%&()’’ilYZacde, *-z!stghjkmn89BC, /:;<qrstfghXPQ2D, =;<xyoprtWOU5679, >?-$%&’()w!WZaOT, ()vwz!’’plnceMN7C, @?;<*-vwxyz!ghjk, >=:fVWIJKLMNRS1A, <’y’’rshikYaeLS49, >=<&)xgkVWHKNOTU, ?*’v!ofnZJPS17BG, /uXbHOPQTU23DEFG, :#&(u’’XYcINP12EF, @?*-&(uvwz!XIN2E.`

16D 80-19a: `123456789ABCDEFG, HIJKLMNOPQRSDEFG, TUVWXYZabcdefgRS, hijklmnofgNOPQFG, pqrsdeLM9ABCDEFG, tuvwpqrsdeJKLMPQ, xyz!’’#vwbc5678BC, $%&’()xyz!’’#Zabc, *-/:()’’#smnoXYMO, ;</:rklmnoMN3478, =<:&’)z!#loWacIS, >?*-hijnUVXYfgNO, </%)!’’uwqjkoegAC, ?=-’)!#uvVWYIS9C, =*-/:moTXYHS2468, >*twrnUXMN3478AB, @UVWXYZabcIR1357, @%&(xy’’TZcHKQREG, ?*pjUYefJQ2358EF.`

16D 80-19b: `123456789ABCDEFG, HIJKLMNO9ABCDEFG, PQRSTUVWXYZaNO78, bcdefghijklmLM56, nopqfghijklmXYZa, rstuvwxyVWKM46FG, z!’’#vwxy12345678, ‘‘#xypqjklmZaJO38, $%&’(uimYaJN28EG,)*-/:’(thmUWYZCD, ;</:!#stwyIJ23BD, :z#swxoqdeglRSTW, =>*-%&’(bcdePQST, :$rsoqceglQRTWAG, ?@><)-/$&(ruHM45, >*%(opflPQSTXYZa, ?@;<)/:$rstuRUVW, *-:’(!’’tvybcPQUW, <)beQSHKLM14569D.`

16D 72-11 4-gon: `34AC1256789BEFGD, DEFGXYZabcdeUVWT, TUVWHJKLMOPQRSNI, IN*-&#$zswnq4AC3,,, fghijklmnopqreSW, stuvwklmnopqrdRV, xyz!uvwqrcOPQVBC, ‘‘#$%hijopbLMNW9A, &’(%twnrZaKN78FG,)’(!gjmpXYJQ56FG, *-)’’xyfjmoHQ12AC.`

16D 72-11 5-gon: 9ABC12345678EFGD, DEFGXYZabcdTUVWe, eWxyz!’’#$Mstuvwr, rstuvwnopqNOmRSl, lmRSfghijkPQABC9,,, HIJKLMNOPQRSTUVW, %&’!’’#$pqtuvwdLV, ()z$oswkbcKS78FG, ()’#nqvjZaJR56FG, *-%&#quhiYIV34BC, *-xy$stfgXHW12BC.

16D 74-13 6-gon: 12FG34569ADE8BC7, 78BC()*’’’#bcghia, abcghiXYZdefklmj, jklmnopqrstuUVWT, TUVWHKLNOQJMPRSI, IJMPRS:-/$vw2FG1,,, vwxyrstulmVWDEFG, z!’’#pqtukmUW9ABC, $%&’defghiNOPQRS, -/*&xyYZcfhi56FG, :()%z!KLMQRS34BC, XZbcefgiHJLMOPQS, notuZbeiklJLOSUV.

16D 76-15 7-gon: imghjVWM7FlXPACk, klXPAC)*-$%S:&’/, /:&’;<’’#b1NO9BGx, xNO9BGdHQ2DEuJLq, quJLnrIKpst3456o, opst3456wcfTUYZ(, (wcfTUYZyvaemghi,,, 123456789ABCDEFG, HIJKLMNOPQBCDEFG, RSTUVWXYZOPQAEFG, abcdefghYZNQ89DG, vwrstumefhUZKL56, yz!’’#$%&’(mefgUY, -wprtujfRUXZKL46, ;<)*z!&’npsuJK36.

7. Section XI

160-21: 123456789ABCDEFGHIJKLMNOPQRSTUVW, 3XY45Za8b9CDEcdFIJKefghiMRjklmTW, nopqrstuvwxyz!’’#9IJK$%&jklm’()*-, /:;<=>?@[\]^_‘{|}~+1+2+3+4+5+6+7+8+9+A+B+C+D+E, +F+G+H+I+J+K+L+M23+N4+O679AB+PDFGJ+QLNOPQSUV, 2+NXY45+O679A+PDEcdIJ+QKL$Q&jklm’SV-, 123XYa8bAB+PDEJ+QK~+R+S+T+3+4+5+U+8+V+W+X+B+C+D+Y, noqruwy’’38CFfiM+Z%O+ajkl+bmS(T)U*+cV, +G+H+d+J+e+K+f+g+NXY+OZab+Pcd+Qefghi+Z$%+ajm’), /+h;+i+j>+k@+l\+m^_+n{+o+F+Gop+d+pqr+J+qv+ewx!+M, nqrvxyz#13+NY457Z8b9ABCDEdFGHIJ+QK, 2679AIJKLMN$%OPQR&jklm’S(T)U*V-W, +r\^+s+n‘|+o2367Za8bACFeN%+ajl(T)U*+cW, +t/+h:+j=+u>+l[+r\+v_+n‘+w+R+S+2+T+3+4+x+U+6+7+8+9+D+y+z, +t/;+!+l[]+m13XY46ZaABCEcdGHM+Z+aR+bT+cW, ~+R+2+3+U+6+’’+#+8+V+A+B+Y+E+y+zfg+ZN$OQRjkS(T)+c-, +u>@+$+l[]+m+v_+n‘{+%|+o1+NY+PcGIJgLi’S)*-, +F+&no+H+’+dstv+e+Kwxy+M+w+R+2+4+5+x+U+6+(+V+A+C+D+)+Y+E, 3XY4589CDEcdFIJKfghLiMQRjklmSTVW, +F+&no+’p+pq+Irs+qtuvwx+Lyz!+M’’#&kl+b(*+c-, +t+h+!+i+j+u+k+$+l+r+m+s+v+n+%+o+w+R+S+T+x+U+’’+#+(+V+W+X+)+Y+y+z.

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