MLADEN PAVIČIĆ*

A THEORY OF DEDUCTION FOR QUANTUM MECHANICS

Abstract. Unified quantum logic which merges all five possible operations of implication in quantum logic is proposed as a theory of deduction for quantum mechanics. This is supported by showing that quantum logic cannot be based on YES-NO experiments carried out on individual quantum systems unless so postulated. As an indication of alternative Kripkean semantics, Dishkant's embedding is carried out with a modal system which is neither reflexive nor symmetric.

1. *Introduction*

The theory to be considered as a candidate for a theory of deduction underlying quantum mechanics in this paper is a particular formulation of quantum logic.

By quantum logic we mean [following Mittelstaedt (1978, 1986), Kalmbach (1974, 1983), Goldblatt (1974), Nishimura (1980), and Dishkant (1974)] an axiomatic calculus for the orthomodular-valid formulas

* Institute for Theoretical Physics, University of Cologne, D-5000 Köln 41, Federal Republic of Germany.

whose syntax is determined by a particular set of axioms and rules of inference, and whose semantics are determined by the equational class of orthomodular lattices on the one hand[1], and by particular modal frames on the other. It should be stressed here that in the literature quantum logic is given many other different meanings as well. For example, it is considered to be an orthomodular partially ordered set with the set of states defined on it[2], simply an orthomodular lattice[3], a row of algebraico-logical structures named quantum propositional logics[4], and finally so-called manuals and semi-Boolean algebras named quantum events logics[5]. None of these structures is, however, in accordance with the usage of "logic" as a deductive theory of valid inferences and consequently, since they cannot serve our purpose, they will no longer concern us.

By a "particular formulation" of quantum logic we mean unified quantum logic (UQL), formulated in Pavičić (1989), which merges all possible operations of implication in quantum logic (definable by other operations) and dispenses with the relation of implication altogether. Unified quantum logic becomes classical logic when the classical implication is used instead of the quantum implications. The axioms and the rules of inference of UQL remain the same in both cases.

The question as to whether quantum logic can be considered a theory of deduction underlying quantum mechanics has been given many contradictory answers to date. It has been argued that quantum logic is necessarily an empirical logic, and that, therefore, it cannot be a theory of a priori valid inferences[6]. At the same time, many axiomatic deductive calculuses, all of which have the equational class of orthomodular lattices as their model, have actually been formulated[7].

We consider it necessary to resolve this controversy before dwelling on deductive quantum logic itself. To this end in Sec. 2 we investigate quantum probability equal to unity which characterizes "predictable" and "repeatable" measurements of the first kind. In this case probability equal to unity ascribes a unique meaning to the ensemble of individual systems measured. It is of course possible to claim, without

[1] Cf. Kalmbach (1983), p. 233.
[2] Gudder (1970).
[3] Nánásiová (1986).
[4] Lock & Hardegree (1985).
[5] Lock & Hardegree (1985a).
[6] For a historical review see Jammer (1974), Ch. 8.6; For an analysis see Mittelstaedt (1986).
[7] For a review see Pavičić (1989).

a contradiction, that probability equal to unity does not ascribe a unique experimental meaning to *all* individual systems as it does to an ensemble of them. Such a claim can, however, be considered as a mathematical peculiarity without a physical meaning. We have, therefore, provided a physically meaningful function which physically differentiates between the two ascriptions although it does not enables an experimental decision between them. Hence, the only way to ascribe an individual system a proper value measured by a repeatable measurement is to *postulate* this.

Thus we come to the conclusion that quantum logic is first of all an "a priori" calculus which is surely weakly confirmable by quantum mechanics in the same way in which classical logic is confirmable by classical mechanics[8]. It may be that quantum logic is strongly empirically confirmable (deducibile) as well, since quantum formalism is in agreement with the both possibilities. However, such an additional feature of individual quantum measurements can hardly be proved by experiments conceivable today.

Whether or not the afore-mentioned a priori axiomatic calculus can be considered a theory of deduction underlying quantum mechanics has encountered scepticism for yet other reasons. First, the objection has been raised that such a calculus does not satisfy many axioms and rules valid in classical logic some of which have traditionally been taken to be indispensable for a "proper" logic[9]. Such an objection has gradually been dropped since many quantum logical deductive systems were actually axiomatized "in a manner completely analogous to classical propositional logic."[10] Secondly, a problem has been raised about the fact that all these deductive systems using different operations of implication could apparently not satisfy a unique axiomatic system[11]. However, we recently provided a unique axiomatization for all five equivalence classes of implications definable in quantum logic by means of other operations (e.g. negation and conjunction)[12]. The system is presented in Sec. 3. And finally, quantum logic is considered to be "intractable"[13] for the lack of simple non-algebraic semantics which are apparently needed to solve certain problems so far unresolved, e.g. as to whether quantum logic has a finite model property or whether it is

[8] See Mittelstaedt (1986).

[9] Jammer (1974); Brody (1984).

[10] Hardegree (1979); For a review see Pavičić (1989).

[11] Zeman (1978), Hardegree (1975, 1981), Georgacarakos (1980).

[12] Pavičić (1987, 1989).

[13] As worded by Goldblatt (1984).

decidable. Although several such semantics have been formulated [Kripkean by Goldblatt (1974) and Dalla Chiara (1986) and probabilistic by Pavičić (1987a)] none of them has proved to be successful in solving these problems. Probabilistic semantics shows that a probability function needed to prove the completeness theorem for the semantics is not guaranteed existence as far as quantum logic proper is concerned. It seems that only by adding particular new axioms, thus obtaining a logic between orthomodular and modular, can we assure the existence of such a function[14]. An analogous conclusion can be conjectured for the Kripkean accessibility relation on the basis of a result obtained by Goldblatt[15] which is based on a reflexive and symmetric relation of accessibility. To this end in Sec. 4 we indicate a possibility of using another relation of accessibility by embedding quantum logic into a modal system which is much weaker than those employed by Dishkant (1977) and Dalla Chiara (1986) although the same translations have been used.

2. *Statistical vs. empirical basis for quantum logic*

In this section we are going to show that quantum logic cannot be based on YES-NO experiments carried out on individual quantum systems unless so postulated.

It has recently been shown that non-discrete observables do not satisfy the repeatability hypothesis[16]. It has also been shown that in the presence of a conservation law even discrete observables are only approximately measurable[17]. Thus we arrive at unsharp reality or generalized quantum mechanics, which boils down to substituting projectors having spectra in {0,1} by effects (unsharp operators/observables) having spectra within [0,1][18]. Analysis of quantum measurements of continous as well as discrete observables along these lines "suggests that we should investigate the approximately repeatable measuring processes as models of measurements in quantum mechanics... not only for continuous observables [but also for] discrete observables [e.g. in the presence of] some conservation law"[19]. This suggestion can also be viewed in the

14 Mayet (1984).
15 Goldblatt (1984).
16 Ozawa (1984).
17 Wigner (1952); Araki & Yanase (1960).
18 For a review see Busch, Grabowski, & Lahti (1989).
19 Ozawa (1984).

light of the fact that both kinds of observables allow a value from their spectra to be a result of a measurement which, however, cannot in general be attributed to a particular property of a measured individual system in a particular state. It is undoubtedly true that eingevalues of a projector *can* be taken to correspond to a particular property of an individual system on which a repeatable YES-NO experiment of the first kind is carried out by simply disregarding the fact that this repeatability is actually a statistical concept[20] However, in the light of the afore-mentioned results it would be interesting to see under which conditions we *can drop* the assumption that an eigenvalue of discrete observables determines a particular property of an individual system, on which a repeatable measurement of the first kind is carried out, without causing any change in the standard Hilbert space description of the measurement. It is well-known that we can do so provided we retain the correspondence between an eigenvalue and a mesured property for an ensemble of individual systems. In other words, for repeatable measurement, i.e. in case when a YES-event is expected to occur with certainty (with probability equal to unity), a NO-event is nevertheless permitted to occur, although with probability zero. This interpretation (often called the statistical interpretation of quantum mechanics) sounds rather artificial and therefore we have recently provided a result [Pavičić (1989a)] according to which the assumption of the repeatability of a measurement carried out on individual quantum systems boils down to an actual jump-discontinuity of a well-defined and experimentally meaningful function for just two end points of a closed interval. Thus, if we want to retain the repeatability assumption for individual systems subjected to measurements of the first order we have to postulate so. A consequence of such a postulate is the above stated discontinuity of a function defined below. While referring the reader to Pavičić (1989a) for details, we shall briefly present but the core of the result.

Let us consider individual quantum systems of spin s prepared, one by one, in a particular spin projection by a preparation (Stern-Gerlach) device along the vertical. Let them be detected by a detection device (another Stern-Gerlach device) deflected at an angle α with respect to the preparation direction. Then quantum mechanics predicts that the probability $p = p(\alpha)$ of confirming the prepared "property" (spin projection) in the long run is given as the square of an appropriate element

[20] von Neumann (1955), pp. 214 & 335.

from the diagonal of the rotation matrix. To simplifiy the argumentation let us take e.g. projection + 1 of spin 1. In this case the probability is given as

$$p = \cos^4 \frac{\alpha}{2} \,.$$

If we now want to treat individual YES-NO measurements, we have to introduce frequencies and if we are to stay within quantum formalism the frequencies have to be introduced on an infinite number of trials, i.e. the frequencies have to be ideal. Let $N_+ = N_+(p,N)$ be the number of successes in the first N trials. Then the frequency with which the measured property is confirmed is N_+/N.

The first basic feature of any YES-NO measurement is that particular individual events are completely independent, and its second basic feature is that trials form an exchangeable sequence. This means that quantum trials are Bernoulli trials and we can obtain, for any probability p defined as $p := \langle N_+/N \rangle$, from open interval $0 < p < 1$, the following two results:

$$P[\lim_{N\to\infty} \frac{N_+(p,N)}{N} = p] = 1 \qquad (1)$$

and

$$\lim_{N\to\infty} P[\frac{N_+(p,N)}{N} = p] = 0 \qquad (2)$$

Eq. (1) means that N_+/N converges to p almost certainly and we abreviate it as $N_+/N \xrightarrow{a.c.} p$. Eq. (2) means that N_+/N acquires a value which is *strictly* equal to p by probability zero which is an interesting characterization of the stochasticity of frequencies of Bernoulli trials.

If we were able to extend Eq. (2) to the closed interval $0 \leqslant p \leqslant 1$ we would immediately obtain the result that the correspondence between eigenvalues and individual properties does not hold since in this case Eq. (2) for $p = 1$ states that $N_+(1,N) = N$ only by probability zero. Of course, that would not contradict Eq. (1) which in this case reads $P[\lim_{N\to\infty}$

$N_+(1,N)/N = 1] = 1$. Unfortunately, the theory of probability cannot say anything about end points of the closed interval. We therefore construct a particular expression which turns out to be a "measure" of the validity of Eq. (2) for $p = 1$ and $p = 0$. The main point leading to the expresion is that angle α is not considered only as a macro-observable measured directly by classical means but also, on the one hand, as a function of frequency $\alpha = \alpha(N_+/N)$ and, on the other hand, as a function of probability $\alpha = \alpha(p)$. The expression is

$$G(p) := L \lim_{N\to\infty} [|\alpha(N_+/N) - \alpha(p)|\, N^{1/2}] = [p(1-p)]^{1/2} \left|\frac{dp}{d\alpha}\right|^{-1}, 0<p<1,$$

where L is an undefined but bounded and non-zero random variable.

Since for our probability $p = \cos^4(\alpha/2)$ we obtain

$$G(p) = \frac{\sin(\alpha/2)[1 + \cos^2(\alpha/2)]^{1/2}}{2\sin(\alpha/2)|\cos(\alpha/2)|}$$

we see that $\lim_{p\to1} G(p) = 2^{-1/2}$ and $\lim_{p\to0} G(p) = \infty$.

If we want to have $N_+(1,N) = N$ and $N_+(0,N) = 0$ we have to assume that $G(p)$ is not continuous at 1 and 0. The consequence is the following.

We can "calculate" angle $\alpha(N_+/N)$ as $2 \cos^{-1}(N_+/N)^{1/4}$ as N approaches infinity since then $N_+/N \overset{a.c.}{\to} p$. We can also "read off" angle $\alpha(p)$ by a macro-instrument since $\alpha(p)$ is the "real" angle corresponding to "objective" probability p. Now, our result simply means that the difference $|\alpha(N_+/N) - \alpha(p)|$ between "microscopically" measured angle $\alpha(N_+/N)$ and "macroscopically" measured angle $\alpha(p) = \alpha$ multipled by $N^{1/2}$, i.e. the expression $|\alpha(N_+/N) - \alpha(p)|N^{1/2}$ never vanishes as N approaches infinity on open interval $0<p<1$. By adopting $N_+(1,N) = N$ and $N_+(0,N) = 0$ we, therefore, cannot but assume that nature differentiates open intervals from closed ones, i.e. distinguishes between two infinitely close points. However, we must admit that no formal reason speaks against such an assumption.

3. Unified quantum logic

Both classical and quantum logic can be considered as logics underlying classical and quantum mechanics in the sense of having the

Boolean and the orthomodular lattice, respectively as their models. This statement can be given the following elaboration.

Classical logic has the set of all subspaces of the phase space of classical particle mechanics which form a Boolean (distributive) lattice as its model, i.e. its Lindenbaum-Tarski algebra. Thus, the operations of the object language of the logic can be interpreted as the set operations and the set relation. Notably, conjunction can be interpreted as set-intersection, disjunction as set-union, negation as set complementation, and finally operation of implication as set-inclusion (the set relation). In other words, the operations from the object language of classical logic find their unique counterparts in the semantic structure of the logic and vice versa, the semantical operations and relation are unique equivalence classes of the operation from the object language of the logic. For, example, the semantical entailment (set inclusion, set relation) boils down to a unique equivalence class of the operation of implication.

Subspaces of the Hilbert space form an orthomodular lattice and an orthomodular lattice is a model for quantum logic, i.e. its Lindenbaum-Tarski algebra. Again, conjunction can be represented as set intersection. Negation is represented as orthocomplement and disjunction as join operation on the lattice. Although semantically nonunique, the latter operations can be considered as syntactically unique. However, in this case there are five different equivalence classes of operations of implication which can all be represented as set-inclusion (set ordering relation)[21]. In the literature such different operations of implication served for formulations of apparently structurally different quantum logics[22].

The purpose of this section is to show that formulations of quantum as well as classical logic that employ different operations of implication are actually not structurally different. Notably, we give a unified axiomatic sustem of both classical and quantum logic employing disjunction, negation, and implication. We named the system unified quantum logic (UQL). If we choose the operation of implication to be any of the five quantum implications defined below we shall have quantum logic. If we choose it to be the classical implication we shall have classical logic.

The system itself is formulated as follows.

[21] Equivalently, we can take negation and implication as syntactically unique in which case there are four (not five; see note 23) different equivalence classes of operations of disjunction.
[22] For a review see Pavičić (1989); See also Georgacarakos (1980).

The propositions are based on elementary propositions p_0, p_1, p_2,... and the following connectives: $\neg$ (negation), $\vee$ (disjunction), and $\rightarrow$ (implication).

The set of propositions $Q°$ is defined formally as follows:

p_j is a proposition for $j = 0, 1, 2, \ldots$

$\neg A$ is a proposition iff A is a proposition.

$A \vee B$ is a proposition iff A and B are propositions.

$A \rightarrow B$ is a proposition iff A and B are propositions.

The conjunction connective is introduced by the following definition:

$A \wedge B := \neg(\neg A \vee \neg B)$

Our metalanguage consists (apart from the common parlance) of axiom schemata from the object language as elementary propositions and of compound metapropositions built up by means of the following metaconnectives: & ("and"), $\veebar$("or"), $\sim$("not"), $\Rightarrow$("if,...then"), $\Leftrightarrow$("iff"), with the usual "classical" meaning.

We define unified quantum logic (UQL) as the axiom system given below. The sign $\vdash$ may be interpreted as "it is asserted in UQL". Connective $\neg$ binds stronger and $\rightarrow$ binds weaker than $\vee$ an $\wedge$, and we shall occasionally omit brackets under the usual convention. To avoid a clumsy statement of the rule of substitution, we use axiom schemata instead of axioms.

Axiom Schemata

A1. $\vdash A \rightarrow A$
A2. $\vdash A \leftrightarrow \neg\neg A$
A3. $\vdash A \rightarrow A \vee B$
A4. $\vdash B \rightarrow A \vee B$
A5. $\vdash B \rightarrow A \vee \neg A$

Rules of Inference

R1. $\vdash A \rightarrow B \ \& \vdash B \rightarrow C \Rightarrow \ \vdash A \rightarrow C$
R2. $\vdash A \rightarrow B \Rightarrow \ \vdash \neg B \rightarrow \neg A$
R3. $\vdash A \rightarrow C \ \& \vdash B \rightarrow C \Rightarrow \ \vdash A \vee B \rightarrow C$
R4. $\vdash A \leftrightarrow B \Rightarrow \ \vdash (C \rightarrow A) \leftrightarrow (C \rightarrow B)$
R5. $\vdash A \leftrightarrow B \Rightarrow \ \vdash (A \rightarrow C) \leftrightarrow (B \rightarrow C)$
R6. $\vdash (A \vee \neg A) \leftrightarrow B \Leftrightarrow \ \vdash B$

where $\vdash A \leftrightarrow B$ means $\vdash A \rightarrow B \ \& \vdash B \rightarrow A$.

The operation of implication $A \rightarrow B$ can be any one belonging to the six classes of equivalence defined by the following six operations of implication:[23]

$A \rightarrow_0 B := \neg A \vee B$ (classical)
$A \rightarrow_1 B := \neg A \vee (A \wedge B)$ (Mittelstaedt)
$A \rightarrow_2 B := \neg B \rightarrow_1 \neg A$ (Dishkant)
$A \rightarrow_3 B := (\neg A \wedge \neg B) \vee (\neg A \wedge B) \vee ((\neg A \vee B) \wedge A)$ (Kalmbach)
$A \rightarrow_4 B := \neg B \rightarrow_3 \neg A$ (non-tollens)
$A \rightarrow_5 B := (A \wedge B) \vee (\neg A \wedge B) \vee (\neg A \wedge \neg B)$ (relevance)

By UQL(i), $i = 0,1,\ldots,5$, we denote UQL in which implication $\rightarrow$ is defined as $\rightarrow_i$, $i = 0,1,\ldots,5$.

We shall prove that UQL(0) is classical logic and that UQL(i), $i = 1,\ldots,5$ are quantum logics by constructing the Lindenbaum-Tarski algebra for each of them and showing that it is a Boolean (i.e. distributive) lattice for classical logic and an orthomodular lattice for quantum logics.

By an orthomodular lattice we mean an algebra $L = \langle L^\circ, \perp, \cap, \cup \rangle$ such that the following conditions are fulfilled for any a, b, c $\in L^\circ$:[24]

L1. $a \cap b = b \cap a$
L2. $(a \cap b) \cap c = a \cap (b \cap c)$
L3. $a^{\perp\perp} = a$
L4. $a \cap a^\perp = 0 \qquad (a \cup a^\perp = 1)$
L5. $a \cap (a \cup b) = a$
L6. $a \cap b = (a^\perp \cup b^\perp)^\perp$
L7. $a \leq b \ \& \ a \cup b^\perp = 1 \Rightarrow b \leq a$

where the *relation* of implication is defined as $a \leq b := a \cap b = a$.

By a Boolean lattice (algebra) we mean an orthomodular lattice which satisfies the following condition[25]:

L8. $a \cup b^\perp = 1 \Rightarrow b \leq a$

[23] For details and further references see Pavičić (1987, 1989); Intead of expressing operations of implication by means of negation and disjunction (conjunction is defined by disjunction and negation) we can correspondingly express disjunction by means of negation and implication in the following *four* ways: $(\neg A \rightarrow \neg B) \rightarrow A$, $(A \rightarrow B) \rightarrow B$, $\neg A \rightarrow (\neg A \rightarrow B)$, and $\neg(A \rightarrow \neg B) \rightarrow B$.

[24] For connections with other defintions which appear in the literature see Zierler (1961), Finch (1969), and Pavičić (1987).

[25] Cf. Finch (1970); In a Boolean lattice (algebra) L7 is redunadant and can be dropped.

In order to stress an equal footing of classical and quantum logic regarding their algebraic semantics and at the same time to avoid clumsy distinction between UQL(0) and UQL(i), $i = 1,\ldots,5$, as well as between Boolean and orthomodular lattices, we shall write UQL in the sequel whenever a particular statement applies to both classical and quantum logics. Furthermore we shall write simply "lattice" meaning a Boolean lattice whenever referring to UQL(0) and an orthomodular one whenever referring to UQL(i), $i = 1,\ldots,5$.

Let us introduce two definitions using this convention.

Definition 1. We call $\mathscr{L} = <L,h>$ a model of the set of formulas Q° (a model of UQL for short) if L is a lattice and if $h\mathrm{:UQL} \mapsto L$ is a morphism in L preserving operations $\neg$, $\vee$, and $\rightarrow$.

Definition 2. We call a proposition $A \in Q^\circ$ true in the model $\mathscr{L}$ if for any morphism h: $\mathrm{UQL} \mapsto L$, $h(A) = 1$ holds.

We can now prove the consistency of UQL for valid formulas from L.

Theorem 1. If $\vdash A$, then A is true in any model of UQL.

Proof. The proof for UQL(0), when the model is a Boolean lattice, is well-known. The proof for UQL(i), $i = 1,\ldots,5$, when the model is an orthomodular lattice, is given in Pavičić (1989). Q.E.D.

To prove the opposite, i.e. the completeness of UQL for the class of valid formulas of L, we first define relation $\equiv$ and give some related lemmas whose proofs are provided in Pavičić (1989).

Definition 3. $A \equiv B \; := \; \vdash A \leftrightarrow B$

Lemma 1. Relation $\equiv$ is a congruence relation on the algebra of propositions $\mathscr{A} = <Q^\circ, \neg, \vee, \rightarrow>$.

Lemma 2. The Lindenbaum-Tarski algebra $\mathscr{A}/\equiv$ is an orthomodular (a Boolean) lattice, i.e. L1-L7 (L1-L8) are true for $\neg/\equiv$, $\vee/\equiv$, and $\rightarrow/\equiv$ turning into the corresponding lattice operations by means of natural morphism $k{:}\mathscr{A} \rightarrow \mathscr{A}/\equiv$.

Corollary. $<\mathscr{A}/\equiv, k>$ is a model of UQL.

Lemma 3. $k(A) = 1 \Rightarrow \; \vdash A$.

Thus we have proved the completeness of UQL for valid formulas of L, i.e. the following theorem.

Theorem 2. If A is true in any model of UQL, then $\vdash A$.

Taken together, UQL is shown to be a proper quantum-logical deductive system as far as its algebraic semantics is concerned. (Perspectives for a modal, i.e. Kripkean semantics for UQL we shall consider in the next section). This also shows that most likely none of the five implications defined above is to be preferred, in contradistinction to conjectures in the literature.

While referring the reader to Pavičić (1989) for details, we shall close this section by just stating that the system possesses a number of desirable properties including the (weak) law of modus ponens, the (weak) law of transitivity, the property of orthomodularity derivable from the axioms of the system, a possibility of the implication to be nested, and a clear formal correspondence with an orthomodular lattice and the operations of implication definable in it.

4. *Modal semantics for quantum logic*

In the previous section we showed that as far as an algebraic semantics is concerned, quantum logic can be considered a proper logic in quite the same sense as classical logic.

Another sense in which an axiomatic system can be considered a proper logic is given by the possibility of finding a particular relation of accessibility which characterizes quantum logic, thus equipping the logic with a modal, i.e. Kripkean semantics. Once found, the relation of accessibility may offer a canonical model would falsify all-theorems, i.e. establish decidability and possibly even the finite model property. For quantum logic such a relation of accessiblity has not been found. What has been achieved is a way of imposing particular restriction on a frame characterizing a weaker, so called orthologic or minimal quantum logic thus obtaining a Kripkean "quasi-semantics" for quantum logic[26]. The relation of accessibility used for this purpose is a reflexive and symmetric one. It determines the orthoframe which characterizes minimal quantum logic. Whether a class of orthoframes characterizes quantum logic proper is not known. What is known is that even if it does, the frames cannot be defined by first-order conditions on such a reflexive and symmetric relation of accessibility. This result prompted its author Goldblatt (1984) to conclude on "intractability" of quantum logic.

However, if it were possible to find a relation of accessibility for quantum logic which is not reflexive and symmetric, then the possibility to impose first order conditions on such a relation in order to characterize the logic would still be open, provided we do not

[26] Goldblatt (1974).

keep too heavily to the (usual) correspondence with the Hilbert space[27].

How to find a relation of accessibility characterizing a logic at all? Obviously, first of all so as to define the relation directly on the logic[28]. But can we find it so as to embed quantum logic into a modal system characterized by the relation? For example, Goldblatt (1974) and Dalla Chiara (1977) embedded minimal quantum logic into the Brouwerian KTB system[29] and Dishkant (1977) embedded quantum logic into an extension of KTB which he designated Br^+. These results were taken as an indication that we can infer metalogical properties of quantum logic from the metalogical properties of a modal logic in which quantum logic can be embedded[30]. On the other hand, it has been shown in Pavičić (1989) that the (extended) KTB, which is characterized by a symmetric and reflexive relation of accessibility is not the only modal system in which (minimal) quantum logic can be embedded. This seems to indicate that we cannot infer the properties of quantum logic directly from the properties of a modal logic in which quantum logic can be embedded. We can only use such modal systems as indicators for possible relations of accessibility which could eventually characterize quantum logic, and for which we have to find out how to define them on quantum logic. However, the translation used in Pavičić (1989) differs from the ones used by Dishkant (1977), Goldblatt (1974), and Dalla Chiara (1977) and we know that by means of different translations we can embed the same logic into different modal systems. For example, by using translation $A^+ := \Box A^+$ classical logic can be embedded into S5 while by using $A^+ := \Box \Diamond A^+$ it can be embedded into S4[31]. Be-

[27] The irreflexive and symmetric orthogonality relation obviously plays a crucial role in an algebraico-logical representation of the Hilbert space quantum formalism. Both Goldblatt's (1974) and Goldblatt's (1984) results were achieved by using this fact; In particular, Goldblatt's (1984) result was achieved by using a direct correspondence with the Hilbert space.

[28] As Goldblatt (1974) and Dishkant (1972) did it for minimal quantum logic. Dishkant (1972) used the condition corresponding to our MA3 which we presented at the end of Sec. 4, together with reflexivity, thus apparently obtaining a stronger result than Goldblatt (1974). However, in proving the completeness Dishkant (1972) did not use the condition corresponding to MA3 but the symmetry, which is a special case of the condition. Actually, according to Goldblatt's result, a relation of accessibility which would fully satisfy the condition, i.e. which would not collapse into a symmetric relation, cannot exist in minimal quantum logic. For quantum logic itself this still has to be proved or disproved.

[29] To call the system "Brouwerian KTB system" is somewhat redundant since the Brouwerian system [called so e.g. by Hughes & Cresswell (1968)] is the KTB system [called so e.g. by Chellas (1980)]. The KTB system is a K system to which T and B axioms are added. T and B are defined at the end of Sec. 4.

[30] See e.g. Dalla Chiara (1986), Sec. 3.

[31] Fitting (1970).

sides, one can argue that the translations used by Dishkant (1977) and Goldblatt (1974) are to be preferred over the one from Pavičić (1989) since the former adopted the classical conjunction: $(A\wedge B)^{+} := A^{+}\wedge B^{+}$ while the latter did not: $(A\wedge B)^{+} := \Box\Diamond(A^{+}\wedge B^{+})$[32]. Therefore, if we wanted to take the possibility of embedding quantum logic into different modal systems as an indication of a new relation of accessibility, which might — if existent on the logic — supersede the old one, we should preferably do this by using the same translation. And this is what we are going to do in this section.

In doing the embedding we shall adopt the translation used by Goldblatt and we shall closely follow the procedure and particular results from Dishkant (1977) and Pavičić (1989). Dishkant (1977) carried out an embedding into a KTB system to which a rule, whose special case is MR2, defined below, was added. He designated this system Br^{+}.

We are going to show that it is possible to embed quantum logic into the system M^{-}, defined below, of which Br^{+} is a proper extension.

We define M^{-} as classical logic, i.e. as UQL(0) (with $A\rightarrow B := \neg A\vee B$) to which the following axiom schemata and rules of inference are added. The sign $\vdash_M$ may be interpreted as "it is asserted in M^{-}". The set of all propositions in M^{-} is denoted as M°. In M^{-}, $\vdash_M \Diamond A \leftrightarrow \neg\Box\neg A$ holds.

Axiom schemata.

MA1. $\vdash_M \Box(A\rightarrow B)\rightarrow(\Box A\rightarrow\Box B)$
MA2. $\vdash_M \Box\Diamond A\rightarrow\Diamond A$
MA3. $\vdash_M \Box A\rightarrow\Box\Diamond\Box A$
MA4. $\vdash_M \Box\Diamond\Box A\rightarrow\Box A$

Rules of inference.

MR1. $\vdash_M A \Rightarrow \vdash_M \Box A$
MR2. $\vdash_M \Box\Diamond A\rightarrow\Box\Diamond B \,\&\, \vdash_M \Box\Diamond B\rightarrow\Diamond A \Rightarrow \vdash_M \Box\Diamond B\rightarrow\Box\Diamond A$

We define the embedding of quantum logic UQL (i), i = 1,...,5 into M^{-} by means of the following translation taken over from Goldblatt (1974).

[32] See the introductory discussion of this section; Notice that the latter translation of the conjuction reduces to the former in both Dishkant's (1977) Br^{+} and in our M^{-} (defined below).

Definition 4.

$p_k^+ := \Box \Diamond q_k \quad (k = 1,2,\ldots)$
$(\neg A)^+ := \Box \neg A^+$
$(A \wedge B)^+ := A^+ \wedge B^+,$

where p_k and q_k are elementary propositions from UQL(i), i = 1,...,5 and M^-, respectively; $\wedge$ in $(A \wedge B)^+$ is the conjunction connective from UQL(i), i = 1,...,5; $\wedge$ in $A^+ \wedge B^+$ is the conjunction connective from M^-, etc.

In order to prove the soundness of the embedding we shall first prove two lemmas and define a relation of equivalence.

Lemma 4. Any $A^+ \in M^\circ$, where $A \in Q^\circ$, is of the form $\Box \Diamond A^\circ$, where $A^\circ \in M^\circ$.

Proof. The proof is carried out by induction on the construction of A. For p_k^+, $p_k^\circ = q_k$; for $(\neg A)^+$, from the induction hypothesis it follows that $(\neg A)^+ = \Box \neg \Box \Diamond A^\circ$ and therefore $(\neg A)^\circ = \neg \Diamond A^\circ$; for $(A \wedge B)^+$, $A^+ \wedge B^+ = \Box \Diamond A^\circ \wedge \Box \Diamond B^\circ = \Box (\Diamond A^\circ \wedge \Diamond B^\circ) = \Box \Diamond \Box (\Diamond A^\circ \wedge \Diamond B^\circ)$ by MA3, MA4, and $\vdash_M \Box (A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$ (which is valid in any normal modal system), and therefore $(A \wedge B)^\circ = \Box (\Diamond A^\circ \wedge \Diamond B^\circ)$. Q.E.D.

We define the equivalence relation on Q° as follows.

Definition 5. $A \equiv B := A^+ \leftrightarrow B^+$.

It can be easily proven that it is really a relation of equivalence, i.e. that it is reflexive and transitive. Also, on account of easily provable theorems from M^-, it follows that it is a relation of congruence. Thus we can consider a natural morphism e: $\mathscr{A} \mapsto \mathscr{A}/\equiv$, where $\mathscr{A}$ is the algebra of propositions (see Sec. 3).

Lemma 5. The algebra $\mathscr{A}/\equiv$ is an orthomodular lattice.

Proof. Let a = e(A), b = e(B), and c = e(C). We have to check Ll-L7 from Sec. 3.

Ad Ll & L2. Obvious.

Ad L3. $\vdash_M A^+ \leftrightarrow A^+$ can be written, according to Lemma 4, as $\vdash_M \Box \Diamond A^\circ \leftrightarrow A^+$ and on the account of MA3 and MA4 we obtain

$\vdash_M \Box\neg\Box\neg A^+ \leftrightarrow A^+$ i.e. $\vdash_M (\neg\neg A)^+ \leftrightarrow A^+$. Hence, $a^{\perp\perp} = a$.

Ad L4. MA2 reads: $\vdash_M \neg\Box\Diamond A^+ \vee \neg\Box\neg A^+$. By Lemma 4, MA3, and MA4 we obtain $\vdash_M (\neg A^+ \vee \neg\Box\neg A^+)$. From this, by using elementary theorems of classical logic we get $\vdash_M (\neg A^+ \vee \neg\Box\neg A^+) \leftrightarrow (\neg B^+ \vee \neg\Box\neg B^+)$. By applying RM: $\vdash_M A \to B \Rightarrow \vdash_M \Box A \to \Box B$, and RM$\Diamond$: $\vdash_M A \to B \Rightarrow \vdash_M \Diamond A \to \Diamond B$, which hold in any normal system, of modal logic we arrive at: $\vdash_M \Box\Diamond(\neg A^+ \vee \neg\Box\neg A^+) \leftrightarrow \Box\Diamond(\neg B^+ \vee \neg\Box\neg B^+)$. From Def. 4, DeMorgan's law gives $(A\vee B)^+ = \Box\Diamond(A^+\vee B^+)$, since R: $\vdash_M (\Box A \wedge \Box B) \leftrightarrow \Box(A\wedge B)$ holds in any normal system[33]. Hence, $a \cup a^\perp = b \cup b^\perp$. Analogously, $a \cap a^\perp = b \cap b^\perp$.

AD L5. $\vdash_M A^+ \wedge \Box\Diamond(A^+\vee B^+) \to A^+$ is obvious. Lemma 4 and elementary theorems of classical logic give $\vdash_M A^+ \to \Box\Diamond A^+ \vee \Box\Diamond B^+$. Since $\vdash_M \Box A \vee \Box B \to \Box(A\vee B)$ is valid in any normal system, this expression can, with the help of theorems from classical logic, be easily reduced to $\vdash_M A^+ \leftrightarrow A^+ \wedge \Box\Diamond(A^+\vee B^+)$. Hence, $a \cap (a \cup b) = a$.

Ad L6. Let us start with $\vdash_M \neg\Diamond(\Box\neg A^+ \vee \Box\neg B^+) \leftrightarrow \Box(\Diamond A^+ \wedge \Diamond B^+)$.

By Lemma 4, MA3, MA4, R (see «Ad L4»), and elementary transformations from classical logic we obtain: $\vdash_M \Box\neg\Box\Diamond(\Box\neg A^+ \vee \Box\neg B^+) \leftrightarrow (A^+\wedge B^+)$. Hence, by Def. 4 and the expression for $(A\vee B)^+$ (see «Ad L4») we get L6.

Ad L7. In order to transform the premises of RM2 we shall use R$\Diamond$: $\vdash \Diamond(A\vee B) \leftrightarrow (\Diamond A \vee \Diamond B)$ and a derivative of it R$\Diamond$': $\vdash_M \Diamond(A \to B) \leftrightarrow (\Box A \to \Diamond B)$[34]. We shall also use MR3: $\vdash_M \Diamond A \to \Diamond B \Leftrightarrow \vdash_M \Box\Diamond A \to \Box\Diamond B$, which is valid in M^-, as can be easily proved by using RM (see «Ad L4») for the right to left direction of the metaimplication and by RM$\Diamond$ (see «Ad L4»), MA3, and MA4 for the other direction. Now, by using elementary trasformations of classical logic we can write MR2 as follows:

MR2'. $\vdash_M \Box\Diamond B \leftrightarrow (\Box\Diamond A \vee \Box\Diamond B)$ & $\vdash_M (\Box\Diamond B \vee \neg\Box\Diamond B) \leftrightarrow (\Diamond A \vee \neg\Box\Diamond B) \Rightarrow \vdash_M \Box\Diamond A \leftrightarrow (\Box\Diamond A \vee \Box\Diamond B)$.

By using R$\Diamond$' we can write MA2 as $\vdash_M \Diamond(\Diamond B \to B)$ and therefore, the second premise of MR2' can be written as $\vdash_M \Diamond(\Box\neg B \vee B) \leftrightarrow \Diamond(A \vee \neg\Box\Diamond B)$.

With the help of MR3, R$\Diamond$, MA3, MA4, Lemma 4, and the rules

[33] The designations RM, RM$\Diamond$, and R are taken over from Chellas (1980), Ch. 4.
[34] See Chellas (1980), pp. 114-123.

from classical logic, and by using MR2' we get:

$\vdash_M B^+ \leftrightarrow \Box\Diamond(A^+ \vee B^+)$ & $\vdash_M \Box\Diamond((\neg B)^+ \vee B^+) \leftrightarrow \Box\Diamond(A^+ \vee (\neg B)^+) \Rightarrow \vdash_M A^+ \leftrightarrow \Box\Diamond(A^+ \vee B^+)$.

Thus, $a \cup b = b$ & $a \cup b^\perp = 1 \Rightarrow a \cup b = a$. Hence, L7 holds, since $a \leqslant b \Leftrightarrow a \cup b = b$ is valid in any ortholattice[35]. Q.E.D.

Since the rest of the proof of the soundness remains the same as in Pavičić (1989), i.e. in Dishkant (1977), we have thereby proved:

Theorem 3. $\vdash A \Rightarrow \vdash_M A^+$.

To prove the completeness of the embedding we shall also refer to Pavičić (1989) and Dishkant (1977).

Theorem 4. $\vdash_M A^+ \Rightarrow \vdash A$

Proof. The proof remains the same as in Pavičić (1989) except that we have to prove MA4 within Lemma 8 of Pavičić (1989). This means that we have to prove $a \Vdash \Box\Diamond\Box A \rightarrow \Box A$. By Lemma 7 of Pavičić we can use (v)-part of Lemma 6 of Dishkant (1977) by taking $[\Box A]$ instead of $[\Box q]$ used there[36]. This proves the lemma and at the same time the theorem. Q.E.D.

Thus we have proved that quantum logic can be embedded in a modal logic which is not characterized by either a reflexive or a symmetric relation of accessibility. Namely, MA2 corresponds to the following condition on the relation of accessibility R[37]:

$$\forall w_1 \exists w_2 [w_1 R w_2 \ \& \ \forall w_3 (w_2 R w_3 \Rightarrow w_1 R w_3)]$$

which is satisfied by any reflexive R, while MA3 corresponds to[38]:

$$\forall w_1 \forall w_2 [w_1 R w_2 \Rightarrow \exists w_3 (w_2 R w_3 \ \& \ \forall w_4 (w_3 R w_4 \Rightarrow w_1 R w_4))]$$

which is satisfied by any symmetric R. On the other hand, T: $\vdash_M \Box A \rightarrow A$ and B: $\vdash_M A \rightarrow \Box\Diamond A$ correspond to reflexivity and symmetry of R respectively. Since no axiom of M^- contains a non-modal proposition and since no rule of inference enables us to infer a non-modal proposition from a modal one, it is obvious that neither T nor B can be inferred in M^-. This proves our claim.

We would like to stress here that it is also possible to embed quantum logic in the system M^- by using Dishkant's (1977) translation instead of Goldblatt's translation, given by Def. 4, as proved in Pavičić (1989b).

[35] An ortholattice is a lattice which satifies the conditions L1-L6. It is meant that $a \cap b = a \Leftrightarrow a \cup b = b$ is satisfied in any ortholattice.

[36] Notice that Dishkant (1977) uses A to designate an elementary proposition.

[37] Lemmon (1977), p. 67.

[38] Hughes & Cresswell (1984), p. 38.

5. *Discussion*

The difficulties one faces when trying to establish semantics for quantum logic, e.g. the absence of a valuation function,the absence of first-order conditions for the usual reflexive and symmetric relation of accessibility, etc., induced many authors to conclude that quantum logic, as opposed to classical logic, is more an empirical calculus for quantum YES-NO measurements than a proper "a priori" logic. However, the result presented in Sec. 2 seems to indicate that apparently just the opposite is the case, i.e. that classical logic is "more empirical" than quantum logic. This claim can receive the following elaboration.

In Sec. 2 we found out that the expression $|\alpha(N_+/N) - \alpha(p)|N^{1/2} = G/L$ never vanishes as N approaches infinity on the open interval $0<p<1$. To be able to compare a classical with a quantum interpretation of the expression, let us take an example for which both interpretations are possible, e.g. electrons subjected to an experiment of the kind descibed in Sec. 2. The quantum interpretation then boils down to the quantum Malus law and the probability function is $p = \cos^2(\alpha/2)$, $0 \leq \alpha \leq \pi$. Therefore, $G_{qm}(p) = 1$, for $0<p<1$. The classical interpretation gives[39] $p = (\pi - \alpha)/\alpha$, $0 \leq \alpha \leq \pi$ Therefore, $G_{cm}(p) = \pi[p(1-p)]^{1/2} = [\alpha(\pi - \alpha)]^{1/2}$, for $0<p<1$, i.e. for $0<\alpha<\pi$. We see that $\lim_{p\to 1} G_{qm}(p) = 1$ as opposed to $\lim_{p\to 1} G_{cm}(p) = 0$.

In other words, quantum logical propositions are not likely to be confirmed by individual YES-NO measurements in a direct way but only by means of their statistics. Quantum logic itself, therefore, turns out to be not an empirical but rather an a priori calculus underlying not quantum measurements themselves but quantum mechanical formalism, which is in turn confirmed by the statistics of quantum YES-NO measurements. Can this calculus, i.e. quantum logic be considered a proper logic?

Quantum logic does satisfy the minimal semantic condition of being a proper logic, that is it has an algebraic semantics. Notably, it has an orthomodular lattice as its model. It was disturbing, however, that is seemed as if five structurally different quantum logics existed due to five different operations of implication. In Sec. 3 we showed, by constructing unified quantum logic, that these logics are not *structurally* different. Actually, their multiplicity only means that the equivalence class of the operation of implication can be expressed in five different

[39] Peres (1978).

ways by means of the operation of disjunction (conjunction) and the operation of negation. Correspondingly, the equivalence class of the operation of disjunction (conjunction) can be expressed in four different ways by means of the operation of implication and the operation of negation.

Thus, if a unique operation of implication is at all needed for the formulation of a "proper" logic underlying quantum mechanical formalism, it can apparently be singled out only by adding some new axioms, perhaps just the ortho-arguesian axiom, to quantum logic. This also means that we perhaps should not rely too heavily on the properties of the Hilbert space while investigating quantum logic further, i.e. while investigating its semantics. In particular, it seems that the modal semantic characterization of quantum logic with the help of a reflexive and symmetric relation of accessibility should be reconsidered. To this end, in Sec. 4 we carried out an embedding of quantum logic in a modal system which is characterized neither by a reflexive nor by a symmetric relation of accessiblity. What remains to be proved or disproved is the existence of the relation on quantum logic and the possibility of characterizing an appropriate frame by means of first-order conditions on the relation.

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A.C.