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Equations, States, and Lattices of Infinite-Dimensional Hilbert Spaces

Norman D. Megill[1] and Mladen Pavičić[2]

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We provide several new results on quantum state space, on the lattice of subspaces of an infinite-dimensional Hilbert space, and on infinite-dimensional Hilbert space equations as well as on connections between them. In particular, we obtain an *n*-variable generalized orthoarguesian equation which holds in any infinite-dimensional Hilbert space. Then we strengthen Godowski's result by showing that in an ortholattice on which strong states are defined, Godowski's equations as well as the orthomodularity hold. We also prove that all six- and four-variable orthoarguesian equations presented in the literature can be reduced to new four- and three-variable ones, respectively, and that Mayet's examples follow from Godowski's equations. To make a breakthrough in testing these massive equations, we designed several novel algorithms for generating Greechie diagrams with an arbitrary number of blocks and atoms (currently testing with up to 50) and for automated checking of equations on them. A way of obtaining complex infinite-dimensional Hilbert space from the Hilbert lattice equipped with several additional conditions and without invoking the notion of state is presented. Possible repercussions of the results on quantum computing problems are discussed.

1. INTRODUCTION

Recent theoretical and experimental developments in the field of quantum computing have opened the possibility of using quantum mechanical states, their superpositions, and operators defined on them, i.e., the Hilbert space formalism, to exponentially speed up computation of various systems, on the one hand, and to simulate quantum systems, on the other. Quantum computers can be looked upon as parallel computing machines. Looking at

[1] Boston Information Group, Belmont, Massachusetts 02478; e-mail: nm@alum.mit.edu; Web page: http://www.shore.net/~ndm/java/mm.html

[2] Department of Physics, University of Maryland, Baltimore County, Baltimore, Maryland 21250, and University of Zagreb, Gradjevinski Fakulet, Kačićeva 26, HR-10000 Zagreb, Croatia; e-mail: mpavicic@faust.irb.hr; Web page: http://m3k.grad.hr/pavicic

the speed of computation, the difference between classical and quantum parallel machines is that in a classical one we increase its speed by increasing its physical space (occupied by electronic components: processors, etc.), while in a quantum one we achieve this by exponentially increasing its state space by means of linearly increased physical space (a register of n quantum bits—*qubits*—prepares a superposition of 2^n states). To make a quantum parallel machine compute a particular problem is tricky and requires a great deal of ingenuity, but a computed system itself need not be quantum and need not be simulated. Actually, all algorithms designed so far are of such a kind. For example, Shor's algorithm [1] factors n-digit numbers, Grover's algorithm [2] searches huge databases, and Boghosian–Taylor's algorithm [3] computes the Schrödinger equation. As opposed to this, a quantum simulator would simulate a quantum system (e.g., an atom, a molecule, . . .) and give its final state directly: a quantum computer working as a quantum simulator would not solve the Scrödinger equation, but would simulate it and the outputs would be its solutions: by typing in a Hamiltonian at a console, we would simulate the system.

Quantum simulation of quantum systems describable in the Hilbert space formalism (by the Schrödinger equation) would only then be possible, however, if we found an algebra underlying Hilbert space in the same way in which the Boolean algebra underlies classical state space. Such an algebra for quantum computers has recently been named *quantum logic* in analogy to the classical logic of classical computers [4]. However, this name is misleading for both types of computers because proper logics, both classical and quantum, have at least two models each [5]. Classical logic has not only a Boolean algebra, but also a nonorthomodular algebra as its model, and quantum logic not only an orthomodular algebra (Hilbert space), but also another nonorthomodular algebra: a weakly orthomodular lattice. What resolves this ambiguity is that as soon as we require either a numerical or a probabilistic evaluation of the propositions of classical logic we are left only with the Boolean algebra [5] and that as soon as we impose probabilistic evaluation (states) on quantum logic we are left only with Hilbert space. Therefore quantum logic itself does not to play a role in the current description of quantum systems. Its standard model—Hilbert space—does.

One can make Hilbert space operational on a quantum computer by imposing lattice equations that hold in any Hilbert space on the computer states using quantum gates. Unfortunately, not very much is known about the equations: explorations of Hilbert space have so far concentrated on operator theory, leaving the theory of the subspaces of a Hilbert space (wherefrom we obtain these equations) virtually unexplored. In this paper we investigate how one can arrive at such equations starting from both algebraic and probabilistic structures of Hilbert space of quantum measurement and computation. We obtain several new results on these structures, give a number of new Hilbert space equations, and systematize, significantly simplify, and mutually reduce already known equations. In Section 2 we give several new characterizations of orthomodularity which we make use of later on. In Section 3 we consider ways in which states can be defined on an ortholattice underlying Hilbert space and make it orthomodular, when quantum, and distributive, when classical. We also analyze several kinds of equations characteristic of strong states in Hilbert space (Godowski's and Mayet's). On the other hand, we present a way of obtaining complex Hilbert space from the Hilbert lattice equipped with several additional conditions, but without invoking the notion of state. In Section 4 we give a new way of presenting orthoarguesian equations and their consequences which must hold in any Hilbert space and for which it is not known whether they are characteristic of the states or not. We reduce the number of variables used in the orthoarguesian-like equations in the literature (from six to four for the standard orthoarguesian equation and to three for all its consequences), we show that all consequences that appear in the literature reduce to a single three-variable equation, and find a new one which does not. In Section 5 we present generalizations of orthoarguesian equations that must hold in any Hilbert space. This previously unknown and unconjectured result is our most important contribution to the theory of infinite-dimensional Hilbert spaces in this paper. In Section 6 we show several distributive properties that must hold in any Hilbert space.

2. ORTHOMODULAR LATTICE UNDERLYING HILBERT SPACE

Closed subspaces of Hilbert space form an algebra called a Hilbert lattice. A Hilbert lattice is a kind of orthomodular lattice which we, in this section, introduce starting with an ortholattice which is a still simpler structure. In any Hilbert lattice the operation *meet*, $a \cap b$, corresponds to set intersection $\mathcal{H}_a \cap \mathcal{H}_b$ of subspaces $\mathcal{H}_a$, $\mathcal{H}_b$ of Hilbert space $\mathcal{H}$, the ordering relation $a \le b$ corresponds to $\mathcal{H}_a \subseteq \mathcal{H}_b$, the operation *join*, $a \cup b$, corresponds to the smallest closed subspace of $\mathcal{H}$ containing $\mathcal{H}_a \cup \mathcal{H}_b$, and the *orthocomplement* a' corresponds to $\mathcal{H}_a^\perp$, the set of vectors orthogonal to all vectors in $\mathcal{H}_a$. Within Hilbert space there is also an operation which has no parallel in the Hilbert lattice: the sum of two subspaces $\mathcal{H}_a + \mathcal{H}_b$, which is defined as the set of sums of vectors from $\mathcal{H}_a$ and $\mathcal{H}_b$. We also have $\mathcal{H}_a + \mathcal{H}_a^\perp = \mathcal{H}$. One can define all the lattice operations on Hilbert space itself following the above definitions ($\mathcal{H}_a \cap \mathcal{H}_b = \mathcal{H}_a \cap \mathcal{H}_b$, etc.). Thus we have $\mathcal{H}_a \cup \mathcal{H}_b = \overline{\mathcal{H}_a + \mathcal{H}_b} = (\mathcal{H}_a + \mathcal{H}_b)^{\perp\perp} = (\mathcal{H}_a^\perp \cap \mathcal{H}_b^\perp)^\perp$ [6, p. 175], where $\overline{\mathcal{H}}_c$ is the closure of $\mathcal{H}_c$, and therefore $\mathcal{H}_a + \mathcal{H}_b \subseteq \mathcal{H}_a \cup \mathcal{H}_b$. When $\mathcal{H}$ is finite dimensional or when the closed subspaces $\mathcal{H}_a$ and $\mathcal{H}_b$ are orthogonal to each other, then $\mathcal{H}_a + \mathcal{H}_b = \mathcal{H}_a \cup \mathcal{H}_b$ [7, pp. 21–29; 8, pp. 66, 67; 9, pp. 8–16].

The projection associated with $\mathcal{H}_a$ is given by $P_a(x) = y$ for vector x from $\mathcal{H}$ that has a unique decomposition $x = y + z$ for y from $\mathcal{H}_a$ and z from $\mathcal{H}_a^{\perp}$. The closed subspace belonging to P is $\mathcal{H}_{\mathrm{P}} = \{x \in \mathcal{H} | P(x) = x\}$. Let $P_a \cap P_b$ denote a projection on $\mathcal{H}_a \cap \mathcal{H}_b$, $P_a \cup P_b$ a projection on $\mathcal{H}_a \cup \mathcal{H}_b$, $P_a + P_b$ a projection on $\mathcal{H}_a + \mathcal{H}_b$ if $\mathcal{H}_a \perp \mathcal{H}_b$, and let $P_a \le P_b$ mean $\mathcal{H}_a \subseteq \mathcal{H}_b$. Then $a \cap b$ corresponds to $P_a \cap P_b = \lim_{n\to\infty}(P_aP_b)^n$ [9, p. 20], a' to $I - P_a$, $a \cup b$ to $P_a \cup P_b = I - \lim_{n\to\infty}[(I - P_a)(I - P_b)]^n$ [9, p. 21], and $a \le b$ to $P_a \le P_b$. Now, $a \le b$ also corresponds to either $P_a = P_aP_b$ or to $P_a = P_bP_a$ or to $P_a - P_b = P_{a\cap b'}$. Two projectors commute iff their associated closed subspaces commute. This means that (see Definition 2.5) $a \cap (a' \cup b) \le b$ corresponds to $P_aP_b = P_bP_a$. In the latter case we have $P_a \cap P_b = P_aP_b$ and $P_a \cup P_b = P_a + P_b - P_aP_b$. We have $a \perp b$, i.e., $P_a \perp P_b$ is characterized by $P_aP_b = 0$ [6, pp. 173–176; 8, pp. 66, 67; 9, pp. 18–21; 10, pp. 47–50].

In this section we give several definitions of an orthomodular lattice, two of which (given by Theorem 2.8) are new. In Section 3 we then show that the orthomodularity of an ortholattice is a consequence of defining strong states on the ortholattice, and in Sections 3 and 4 we show that it also a consequence of other more restrictive lattice conditions: Godowski equations and orthoarguesian equations.

Definition 2.1. An ortholattice (OL) is an algebra $\langle \mathcal{L}_{\mathrm{o}}, ', \cap, \cup \rangle$ such that the following conditions are satisfied for any $a, b, c, d, e, f, g, h \in \mathcal{L}_{\mathrm{o}}$:

$$(b \cap (c \cap a)) \cup a = a \tag{2.1}$$

$$((a \cap (b \cap (f \cup c))) \cup d) \cup e = ((((g \cap g') \cup (c' \cap f')')' \cap (a \cap b)) \cup e) \cup ((h \cup d) \cap d) \tag{2.2}$$

Lemma 2.2. The following conditions hold in any OL: $a \cup b = b \cup a$, $(a \cup b) \cup c = a \cup (b \cup c)$, $a'' = a$, $a \cup (a \cap b) = a$, $a \cap b = (a' \cup b')'$. Also, an algebra in which these conditions hold is an OL.

Proof. As given in ref. 11. ■

Definition 2.3. An orthomodular lattice (OML) is an ortholattice in which any one of the following holds:

$$a \equiv_i b = 1 \Rightarrow a = b, \qquad i = 1, \ldots, 5 \tag{2.3}$$

where $a \equiv_i b \overset{\mathrm{def}}{=} (a \to_i b) \cap (b \to_0 a)$, $i = 1, \ldots, 5$, where $a \to_0 b \overset{\mathrm{def}}{=} a' \cup b$, $a \to_1 b \overset{\mathrm{def}}{=} a' \cup (a \cap b)$, $a \to_2 b \overset{\mathrm{def}}{=} b' \to_1 a'$, $a \to_3 b \overset{\mathrm{def}}{=} (a' \cap b) \cup$

$(a' \cap b') \cup (a \to_1 b)$, $a \to_4 b \overset{\text{def}}{=} b' \to_3 a'$, and $a \to_5 b \overset{\text{def}}{=} (a \cap b) \cup (a' \cap b) \cup (a' \cap b')$.

The equivalence of this definition to the other definitions in the literature follows from Lemma 2.1 and Theorem 2.2 of ref. 5 and the fact that Eq. (2.3) fails in lattice O6 (Fig. 1a), meaning it implies the orthomodular law by Theorem 2 of ref. 8, p. 22.

Definition 2.4. We have

$$a \equiv b \overset{\text{def}}{=} (a \cap b) \cup (a' \cap b') \tag{2.4}$$

We note that $a \equiv b = a \equiv_5 b$ holds in all OMLs, so these two identities may be viewed as alternate definitions for the same operation in OMLs. The equality also holds in lattice O6, so they may be used interchangeably in any orthomodular law equivalent added to ortholattices; in particular $\equiv$ may be substituted for $\equiv_5$ in the $i = 5$ case of Eq. (2.3). However, $a \equiv b = a \equiv_5 b$ does not hold in all ortholattices as shown in ref. 5, so the two identities should be considered to be different operations from an ortholattice point of view.

Definition 2.5. We say that a and b *commute* in OML and write aCb when either of the following holds [12, 13].

$$a = (a \cap b) \cup (a \cap b') \tag{2.5}$$

$$a \cap (a' \cup b) \le b \tag{2.6}$$

Lemma 2.6. An OL in which (2.5) and (2.6) follow from each other is an OML.

Yet other forms of the orthomodularity condition are the following ones.

Lemma 2.7. An OL in which any one of the following conditions holds is an OML and vice versa:

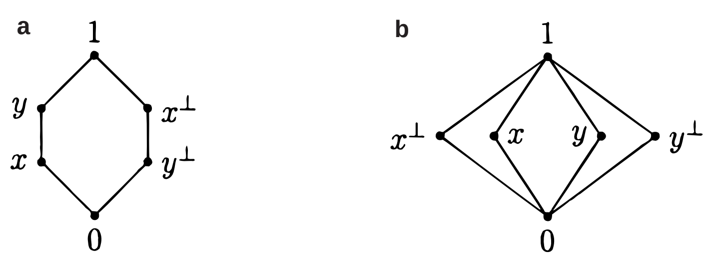

Fig. 1. (a) Lattice O6. (b) Lattice MO2.

$$a \to_i b = 1 \Leftrightarrow a \le b, \qquad i = 1, \ldots, 5 \tag{2.7}$$

Proof. The proof of Eq. (2.7) is given in refs. 14 and 15. We stress that the $\Leftarrow$ direction holds in any OL. ∎

Later we use a definition based on the following "transitivity" theorem, which does not work for $\equiv_i$, $i = 1, \ldots, 4$. Note that in any OML, $a \equiv b = (a \cap b) \cup (a' \cap b') = (a \to_i b) \cap (b \to_i a)$ for $i = 1, \ldots, 5$ and that instead of the Definition 2.3, one can use one with $a \equiv_1 b = (a \to_1 b) \cap (b \to_4 a)$, etc. [5]

Theorem 2.8. An ortholattice in which

$$(a \equiv b) \cap (b \equiv c) \le a \equiv c \tag{2.8}$$

$$(a \equiv b) \cap (b \equiv c) = (a \equiv b) \cap (a \equiv c) \tag{2.9}$$

hold is an orthomodular lattice and vice versa.

The same statement holds for $(a \to_i b) \cap (b \to_i a)$, $i = 1, \ldots, 5$, being substituted for $a \equiv b$.

Proof. Equations (2.8) and (2.9) fail in lattice O6, so they imply the orthomodular law.

For the converse with Eq. (2.8), we start with $((a \cap b) \cup (a' \cap b')) \cap (b' \cup (b \cap c))$. It is easy to show that $(a' \cap b')C(b' \cup (b \cap c))$ and $(a' \cap b')C(a \cup b)$. By applying the Foulis–Holland (FH) theorem [12] to our starting expression, we obtain $((a \cap b) \cap (b' \cup (b \cap c))) \cup ((a' \cap b') \cap (b' \cup (b \cap c)))$. The first conjunction is by orthomodularity equal to $a \cap b \cap c$. The disjunction is thus equal to or less than $a' \cup (a \cap c)$ and we arrive at $(a \equiv b) \cap (b \to_1 c) \le (a \to_1 c)$. By multiplying both sides by $(c \to_1 b)$ we get $(a \equiv b) \cap (b \equiv c) \le (c \to_1 b) \cap (a \to_1 c) \le (a \to_1 c)$. By symmetry we also have $(a \equiv b) \cap (b \equiv c) \le (c \to_1 a)$. A combination of the latter two equations proves the theorem. We draw the reader's attention to the fact that $(a \to_1 b) \cap (b \equiv c) \le (a \to_1 c)$ does not hold in all OMLs (it is violated by MO2). For the converse with Eq. (2.9) we start with Eq. (2.8) and obtain $(a \equiv b) \cap (b \equiv c) \le (a \equiv b) \cap (a \equiv c)$. On the other hand, starting with $(a \equiv b) \cap (a \equiv c) \le (b \equiv c)$, we obtain $(a \equiv b) \cap (b \equiv c) \le (a \equiv b) \cap (a \equiv c)$. Therefore the conclusion.

As for the statements with $(a \to_i b) \cap (b \to_i a)$, $i = 1, \ldots, 5$, substituted for $a \equiv b$, they fail in O6, so they imply the orthomodular law. For the converse it is sufficient to note that in any OML the following holds: $(a \to_i b) \cap (b \to_i a) = a \equiv b$, $i = 1, \ldots, 5$. ∎

We conclude this section with an intriguing open problem whose partial solutions we find with the help of states defined on OML in the next section.

Theorem 2.9. In any OML the following conditions follow from each other:

$$(a \equiv b) \cap ((b \equiv c) \cup (a \equiv c)) \le (a \equiv c) \tag{2.10}$$

$$(a \equiv b) \to_0 ((a \equiv c) \equiv (b \equiv c)) = 1 \tag{2.11}$$

$$(a \equiv b) \cap ((b \equiv c) \cup (a \equiv c)) = ((a \equiv b) \cap (b \equiv c)) \cup ((a \equiv b) \cap (a \equiv c)) \tag{2.12}$$

The relation (2.10) fails in O6; Eqs. (2.11) and (2.12) fail in lattices in which weakly OML (WOML) fail [16], but do not fail in O6.

Proof. To obtain Eq. (2.11) from (2.10), we apply Lemma 2.7 ($i = 1$): $1 = (a \equiv b)' \cup ((b \equiv c)' \cap (a \equiv c)') \cup (((a \equiv c) \cap (a \equiv b)) \cap ((b \equiv c) \cup (a \equiv c))) =$ [Eq. (2.9)] $= (a \equiv b)' \cup ((a \equiv c) \cap (b \equiv c)) \cup ((a \equiv c)' \cap (b \equiv c)')$. Reversing the steps yields (2.10) from (2.11).

To get Eq. (2.12), we first note that one can easily derive $(a \equiv b) \cap ((b \equiv c) \cup (a \equiv c)) = (a \equiv c) \cap (a \equiv b)$ from (2.10). Then one gets (2.12) by applying Eq. (2.9).

To arrive at (2.10) starting from Eq. (2.12), we apply Eq. (2.9) and reduce the right-hand side of Eq. (2.12) to $(a \equiv b) \cap (a \equiv c)$, which yields (2.10). ■

An open problem is whether conditions (2.11) and (2.12) hold in any WOML and whether these conditions together with condition (2.10) hold in any OML. Note that Eq. (2.10) fails in O6 only because Eqs. (2.9) and (2.8), which we used to infer it from Eq. (2.12), fail in O6. We scanned all available orthomodular Greechie lattices[3] with up to 14 blocks (without legs and with 3 atoms in a block; this makes 271,930 legless lattices), over 400,000 lattices with up to 17 blocks, and selected lattices with up to 38 blocks, but there was no violation of any of them by (2.10), so there is a strong indication that these conditions might hold in any OML, but we were not able to prove or disprove this. We think it is an intriguing problem because repeated attempts to prove these conditions in WOML, OML, or Hilbert space always brought us to a kind of a vicious circle and also because we were unable to prove that an even weaker condition holds in any OML. We have, however, proved that the latter condition holds in Hilbert space and we give the proof in the next section [Eq. (3.30)].

[3] We obtain the Greechie lattices with practically arbitrary number of atoms and blocks by using the technique of *isomorph-free exhaustive generation* [17]. The reader can retrieve many lattices with up to 38 atoms and blocks at ftp://cs.anu.edu.au/pub/people/bdm/nauty/greechie.html and ftp://m3k.grad.hr/pavicic/greechie/diagrams (legless), and a program for making any desired set of lattices written in C by B.D. McKay at ftp://m3k.grad.hr/pavicic/greechie/program.

3. STATES AND THEIR EQUATIONS

In the standard approach of reconstructing Hilbert space one starts from an orthomodular lattice (OML) then defines a state on the OML and imposes additional conditions on the state as well on the OML to eventually arrive at the Hilbert space representation of such a mixed lattice–state structure. To get an insight into the latter structure, below we first define a state on a lattice and pinpoint a difference between classical and quantum strong states (Definitions 3.1 and 3.1 and Theorems 3.3, 3.8, and 3.10).

Alternatively, one can reconstruct Hilbert space solely by means of the lattice theory. We start with an ortholattice (OL), build the Hilbert lattice (Definition 3.4 and Theorem 3.5), and with the help of three additional axioms arrive at its complex Hilbert space representation (Theorem 3.6) without invoking the notion of state at all. The states needed for obtaining mean values of measured observables follow from Gleason's theorem.

Going back to the traditional approach, we explore how far one can go in reconstructing the Hilbert space starting with a strong state defined on OL without invoking any further lattice or state condition. We show that strong quantum states imposed on OL turn the latter into an OML in which the so-called Godowski equations hold and obtain several new traits of the equations and Greechie lattices much simpler than the original ones to characterize them (Theorems and Lemmas 3.10–3.19 and 3.21–3.23). In the end we derive Mayet's equations from Godowski's (Theorem 3.20).

Definition 3.1. A state on a lattice L is a function m: $L \to [0, 1]$ (for real interval [0, 1]) such that $m\,(1) = 1$ and $a \perp b \Rightarrow m\,(a \cup b) = m(a) + m(b)$, where $a \perp b$ means $a \le b'$.

This implies $m(a) + m(a') = 1$ and $a \le b \Rightarrow m(a) \le m(b)$.

Definition 3.2. A nonempty set S of states on L is called a strong set of *classical* states if

$$(\exists m \in S)(\forall a, b \in \mathrm{L})((m(a) = 1 \Rightarrow m(b) = 1) \Rightarrow a \le b) \qquad (3.1)$$

and a strong set of *quantum* states if

$$(\forall a, b \in \mathrm{L})(\exists m \in S)((m(a(= 1 \Rightarrow m(b) = 1) \Rightarrow a \le b) \qquad (3.2)$$

We assume that L contains more than one element and that an empty set of states is not strong. Whenever we omit the word "quantum" we mean condition (3.2).

We have not seen the first part of Definition 3.2 in the literature, but consider it worth defining it because of the following theorem.

Theorem 3.3. Any ortholattice that admits a strong set of classical states is distributive.

Proof. Condition (3.2) follows from (3.1), and by Theorem 3.10 an ortholattice that admits a strong set of classical states is orthomodular. Let now a and b be any two lattice elements. Assume, for state m, that $m(b) = 1$. Since the lattice admits a strong set of classical states, this implies $b = 1$, so $m(a \cap b) = m(a \cap 1) = m(a)$. But $m(a') + m(a) = 1$ for any state, so $m(a \to_1 b) = m(a') + m(a \cap b) = 1$. Hence we have $m(b) = 1 \Rightarrow m(a \to_1 b) = 1$, which means (since the ortholattice admits a strong set of classical states) that $b \le a \to_1 b$. This is another way of saying aCb [13]. By FH, an orthomodular lattice in which any two elements commute is distributive. ∎

We see that that a description of any classical measurement by a classical logic (more precisely by its lattice model, a Boolean algebra) and by a classical probability theory coincide because we can always find a single state (probability measure) for all lattice elements. As opposed to this, a description of any quantum measurement consists of two inseparable parts, a quantum logic (i.e., its lattice model, an orthomodular lattice) and a quantum probability theory, because we must obtain different states for different lattice elements.

In order to enable an isomorphism between an orthocomplemented orthomodular lattice and the corresponding Hilbert space we have to add further conditions to the lattice. These conditions correspond to the essential properties of any quantum system such as superposition and make the so-called Hilbert lattice as follows [12, 18]:

Definition 3.4. An OML which satisfies the following conditions is a Hilbert lattice (HL):

1. *Completeness*: The meet and join of any subset of an HL always exist.
2. *Atomic*: Every nonzero element in an HL is greater than or equal to an atom. (An atom a is a nonzero lattice element with $0 < b \le a$ only if $b = a$.)
3. *Superposition Principle*: (The atom c is a superposition of the atoms a and b if $c \ne a$, $c \ne b$, and $c \le a \cup b$.) (a) Given two different atoms a and b, there is at least one other atom c, $c \ne a$ and $c \ne b$, that is a superposition of a and b. (b) If the atom c is a superposition of distinct atoms a and b, then atom a is a superposition of atoms b and c.
4. *Minimal length*: The lattice contains at least three elements a, b, c satisfying $0 < a < b < c < 1$.

Note that atoms correspond to pure states when defined on the lattice. We recall that the *irreducibility* and the *covering property* follow from the superposition principle [19, pp. 166, 167]. We also recall that any Hilbert lattice must contain a countably infinite number of atoms [18]. The above conditions suffice to establish isomorphism between HL and the closed subspaces of any Hilbert space, $\mathcal{C}(\mathcal{H})$, through the following well-known theorem [20, §§33, 34]:

Theorem 3.5. For every Hilbert lattice HL there exists a field $\mathcal{K}$ and a Hilbert space $\mathcal{H}$ over $\mathcal{K}$ such that $\mathcal{C}(\mathcal{H})$ is orthoisomorphic to HL.

Conversely, let $\mathcal{H}$ be an infinite-dimensional Hilbert space over a field $\mathcal{K}$ and let

$$\mathcal{C}(\mathcal{X}) \stackrel{\text{def}}{=} \{\mathcal{X} \subseteq \mathcal{H} | \mathcal{X}^{\perp\perp} = \mathcal{H}\} \tag{3.3}$$

be the set of all biorthogonal closed subspaces of $\mathcal{H}$. Then $\mathcal{C}(\mathcal{H})$ is a Hilbert lattice relative to

$$a \cap b = \mathcal{X}_a \cap \mathcal{X}_b \qquad \text{and} \qquad a \cup b = (\mathcal{X}_a + \mathcal{X}_b)^{\perp\perp} \tag{3.4}$$

In order to determine the field over which Hilbert space in Theorem 3.5 is defined, we use the following theorem.

Theorem 3.6. [Solèr–Mayet–Holland] Hilbert space $\mathcal{H}$ from Theorem 3.5 is an infinite-dimensional one defined over a complex field $\mathbb{C}$ if the following conditions are met:

5. *Infinite orthogonality*: Any HL contains a countably infinite sequence of orthogonal elements [21].
6. *Unitary orthoautomorphism*: For any two orthogonal atoms a and b there is an automorphism $\mathcal{U}$ such that $\mathcal{U}(a) = b$, which satisfies $\mathcal{U}(a') = \mathcal{U}(a)'$, i.e., it is an *orthoautomorphism*, and whose mapping into $\mathcal{H}$ is a unitary operator U, and therefore we also call it *unitary* [12].
7. $\mathbb{C}$ *characterization*: There are pairwise orthogonal elements a, b, $c \in$ L such that $(\exists d, e \in \mathrm{L})\ (0 < d < a \ \&\ 0 < e < b)$ and there is an automorphism $\mathcal{V}$ in L such that $(\mathcal{V}(c) < c)$, $(\forall f \in L\colon f \le a)(\mathcal{V}(f) = f)$, $(\forall g \in \mathrm{L}\colon g \le b)(\mathcal{V}(g) = g)$, and $(\exists h \in L)\ (0 \le h \le a \cup b \ \&\ \mathcal{V}(\mathcal{V}(h)) \ne h)$ [22].

Proof [12]. By Theorem 3.5, to any two orthogonal atoms a and b there correspond orthogonal one-dimensional subspaces (vectors) e and f from $\mathcal{H}$ such that $a = \mathcal{K}e$ and $b = \mathcal{K}f$. The unitary orthoautomorphism $\mathcal{U}$ maps into the unitary operator U so as to give $U(e) = \alpha f$ for some $\alpha \in \mathcal{K}$. From this and from the unitarity of U we get $\langle e, e\rangle = \langle U(e), U(e)\rangle = \langle \alpha f, \alpha f\rangle = \alpha\langle f, f\rangle\alpha^*$. Hence, there is an infinite orthogonal sequence $\{e_i: i = 1, 2, \ldots\}$ such that $\langle e_i, e_i\rangle = \langle f_j, f_j\rangle$ for all i, j. Then Solèr's [21] and Mayet's [22] theorems prove the claim. ■

We have seen that the definition of the "unitarity" of the *unitary automorphism* in the previous theorem is not given directly in HL, but through the inner product of the corresponding Hilbert space (whose existence is guaranteed by Theorem 3.5). A pure lattice version of the definition of the *unitary automorphism* formulated by Holland [12] is not known, but it is known that it can be replaced by Morash's purely lattice-theoretic *angle bisecting* condition in HL [21].

From the previous two theorems we see that to arrive at the basic Hilbert space structure we do not need the notion of state, i.e., of the probability of geting a value of a measured observable. This probability and state follow uniquely from Hilbert space by Gleason's theorem and we can use them to make probabilistic (the only available ones in the quantum theory) predictions of an observable $\mathcal{A}$: $Prob(\mathcal{A}) = tr(\rho\mathcal{A})$, there tr is the trace and ρ is a density matrix [6, p. 178]. Alternatively, we can start with the pure states that correspond to one-dimensional subspaces of Hilbert space, i.e., to vectors of Hilbert space and to atoms in the Hilbert lattice.

Definition 3.7. A state m is called *pure* if, for all states m_1, m_2 and all reals $0 < \lambda < 1$, the equality $m = \lambda m_1 + (1 - \lambda)m_2$ implies $m = m_1 = m_2$.

According to Gleason's theorem [23], for every vector $\Psi_m \in \mathcal{H}$, $\|\Psi_m\| = 1$ and for every P_a, where P_a is a projector on the subspace $\mathcal{H}_a$, there exists a unique inner product $\langle P_a\Psi_m, \Psi_m\rangle$ which is a pure state $m(a)$ on $\mathcal{C}(\mathcal{H})$. By the spectral theorem, to each subspace there corresponds a self-adjoint operator $\mathcal{A}$, and we write $P_a = P_{\mathcal{A}}$. The mean value of $\mathcal{A}$ in the state m is $\langle\mathcal{A}\rangle = \mathrm{Exp}\ m(\mathcal{A}) = \int \alpha d\ \langle P_{\mathcal{A},\{\alpha\}}\Psi_m, \Psi_m\rangle = \langle\mathcal{A}\Psi_m, \Psi_m\rangle$ [24].

So, Conditions 1–4 of Definition 3.4 and 5–7 from Definition 3.6 enable a one-to-one correspondence between the lattice elements and the closed subspaces of the infinite-dimensional Hilbert space of a quantum system and Gleason's theorem enables a one-to-one correspondence between states and mean values of the operators measured on the system provided the above strong states (probability measures) are defined on them. The usage of strong states here is somewhat unusual because most authors use full states instead [12, 24, 8]. To prove that the correspondence (isomorphism) holds for the strong states as well, we only have to prove that Hilbert space admits strong states because the other direction follows from the fact that any strong set of states is full. The result is not new (it appears, e.g., in ref. 19, p. 144), but we give here a proof communicated to us by René Mayet, for the sake of completeness.

Theorem 3.8. Any Hilbert lattice admits a strong set of states.

Proof. We need only to use pure states defined by unit vectors: If a and b are closed subspaces of a Hilbert space $\mathcal{H}$ such that a is not contained in b, there is a unit vector u of $\mathcal{H}$ belonging to $a - b$. If for each c in the lattice of all closed subspaces of $\mathcal{H}$, $\mathcal{C}(\mathcal{H})$, we define $m(c)$ as the square of the norm of the projection of u onto c, then m is a state on $\mathcal{H}$ such that $m(a) = 1$ and $m(b) < 1$. This proves that $\mathcal{C}(\mathcal{H})$ admits a strong set of states, and this proof works in each of the three cases where the underlying field is the field of real numbers, of complex numbers, or of quaternions.

We can formalize the proof as follows:

$$(\forall a, b \in L)((\sim a \le b) \Rightarrow (\exists m \in S)(m(a) = 1 \;\&\; \sim m(b) = 1))$$
$$\Rightarrow (\forall a, b \in L)(\exists m \in S)((m(a) = 1 \Rightarrow m(b) = 1) \Rightarrow a \le b) \quad \blacksquare$$

So, any Hilbert space admits strong states and we need them to predict outcomes of measurements. But there is more to it: states, when defined on an ortholattice, impose very strong conditions on it. In particular, they impose a class of orthomodular equations which hold in $\mathcal{C}(\mathcal{H})$ and do not hold in all OMLs: Godowski's [25] and Mayet's [26] equations. In the rest of this section we first give some alternative formulations of Godowski's equations and present a new class of lattices in which the equations fail. Then we show that Mayet's Examples 2–4, which were meant to illustrate a generalization of Godowski's equations, are nothing but special cases of the latter equations.

Definition 3.9. Let us call the following expression the *Godowski identity*:

$$a_1 \overset{\gamma}{\equiv} a_n \overset{\text{def}}{=} (a_1 \to_1 a_2) \cap (a_2 \to_1 a_3)$$
$$\cdots \cap (a_{n-1} \to_1 a_n) \cap (a_n \to_1 a_1), \qquad n = 3, 4, 5, \ldots \tag{3.5}$$

We define $a_n \overset{\gamma}{\equiv} a_1$ in the same way with variables a_i and a_{n-i+1} swapped; in general $a_i \overset{\gamma}{\equiv} a_j$ will be an expression with $|j - i| + 1 \ge 3$ variables $a_i, \ldots, a_j$ first appearing in that order. For completeness and later use (Theorem 3.22) we define $a_i \overset{\gamma}{\equiv} a_i \overset{\text{def}}{=} (a_i \to_1 a_i) = 1$ and $a_i \overset{\gamma}{\equiv} a_{i+1} \overset{\text{def}}{=} (a_i \to_1 a_{i+1}) \cap (a_{i+1} \to_1 a_i) = a_i \equiv a_{i+1}$, the last equality holding in any OML. We also define $a_1 \overset{\delta}{\equiv} a_n$, etc., with the substitution of $\to_2$ for $\to_1$ in $a_1 \overset{\gamma}{\equiv} a_n$, etc.

Theorem 3.10. Godowski's equations [25]

$$a_1 \overset{\gamma}{\equiv} a_3 = a_3 \overset{\gamma}{\equiv} a_1 \tag{3.6}$$

$$a_1 \overset{\gamma}{\equiv} a_4 = a_4 \overset{\gamma}{\equiv} a_1 \tag{3.7}$$

$$a_1 \overset{\gamma}{\equiv} a_5 = a_5 \overset{\gamma}{\equiv} a_1 \tag{3.8}$$
$$\cdots$$

hold in all ortholattices (OLs) with strong sets of states. An OL to which these equations are added is a variety smaller than OML.

We shall call these equations n-Go (3-Go, 4-Go, etc.). We also denote by nGO (3GO, 4GO, etc.) the OL variety determined by n-Go (which we also call the nGO law).

Proof. The proof is similar to that in ref. 25. By Definition 3.1 we have $m(a_1 \to_1 a_2) = m(a_1') + m(a_1 \cap a_2)$, etc., because $a_1' \le (a_1' \cup a_2')$, i.e., $a_1' \perp (a_1 \cap a_2)$ in any ortholattice. Assuming $m(a_1 \overset{\gamma}{\equiv} a_n) = 1$, we get $m(a_1 \to_1 a_2) = \cdots = m(a_{n-1} \to_1 a_n) = m(a_n \to_1 a_1) = 1$. Hence, $n = m(a_1 \to_1 a_2) \cdots + m(a_{n-1} \to_1 a_n) + m(a_n \to_1 a_1) = m(a_n \to_1 a_{n-1}) \cdots + m(a_2 \to_1 a_1) + m(a_1 \to_1 a_n)$. Therefore, $m(a_n \to_1 a_{n-1}) = \cdots = m(a_2 \to_1 a_1) = m(a_1 \to_1 a_n) = 1$. Thus, by Definition 3.2 for strong quantum states, we obtain $(a_1 \overset{\gamma}{\equiv} a_n) \le (a_n \to_1 a_{n-1}), \ldots, (a_1 \overset{\gamma}{\equiv} a_n) \le (a_2 \to_1 a_1)$, and $(a_1 \overset{\gamma}{\equiv} a_n) \le (a_1 \to_1 a_n)$, wherefrom we get $(a_1 \overset{\gamma}{\equiv} a_n) \le (a_n \overset{\gamma}{\equiv} a_1)$. By symmetry, we get $(a_n \overset{\gamma}{\equiv} a_1) \le (a_1 \overset{\gamma}{\equiv} a_n)$. Thus $(a_1 \overset{\gamma}{\equiv} a_n) = (a_n \overset{\gamma}{\equiv} a_1)$.

nGO is orthomodular because 3-Go fails in O6, and n-Go implies $(n-1)$-Go in any OL (Lemma 3.17). It is a variety smaller than OML because 3-Go fails in the Greechie lattice, from Fig. 2a ■

The following lemma provides a result we will need.

Lemma 3.11. The following equation holds in all OMLs:

$$(a_1 \equiv a_2) \cdots \cap (a_{n-1} \equiv a_n) = (a_1 \cdots \cap a_n) \cup (a_1' \cdots \cap a_n'), \qquad n \ge 2 \tag{3.9}$$

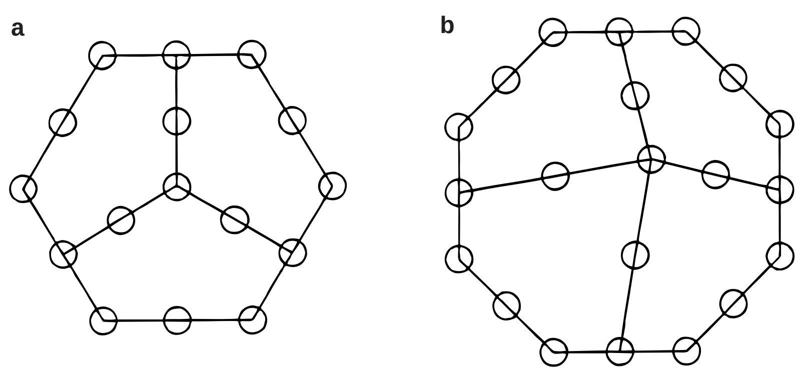

Fig. 2. Greechie diagrams for (a) OML G3 and (b) OML G4.

Proof. We use induction on n. The basis is simply the definition of $\equiv$. Suppose $(a_1 \equiv a_2) \cdots \cap (a_{n-2} \equiv a_{n-1}) = (a_1 \cdots \cap a_{n-1}) \cup (a_1' \cdots \cap a_{n-1}')$. Multiplying both sides by $a_{n-1} \equiv a_n = (a_{n-1} \to_1 a_n) \cap (a_n \to_2 a_{n-1})$, we have

$$\begin{aligned}
&(a_1 \equiv a_2) \cdots \cap (a_{n-1} \equiv a_n)\\
&\quad = [((a_1 \cdots \cap a_{n-1}) \cup (a_1' \cdots \cap a_{n-1}'))\\
&\qquad \cap (a_{n-1} \to_1 a_n)] \cap (a_n \to_2 a_{n-1})\\
&\quad = [(a_1 \cdots \cap a_n) \cup (a_1' \cdots \cap a_{n-1}')] \cap (a_n \to_2 a_{n-1})\\
&\quad = (a_1 \cdots \cap a_n) \cup (a_1' \cdots \cap a_n')
\end{aligned}$$

FH was used in the last two steps, whose details we leave to the reader. ■

Theorem 3.12. An OL in which any of the following equations holds is an nGO and vice versa:

$$a_1 \overset{\delta}{\equiv} a_n = a_n \overset{\delta}{\equiv} a_1 \tag{3.10}$$

$$\begin{aligned} a_1 \overset{\gamma}{\equiv} a_n = {} & (a_1 \equiv a_2) \cap (a_2 \equiv a_3)\\ & \cdots \cap (a_{n-1} \equiv a_n) \end{aligned} \tag{3.11}$$

$$\begin{aligned} a_1 \overset{\delta}{\equiv} a_n = {} & (a_1 \equiv a_2) \cap (a_2 \equiv a_3)\\ & \cdots \cap (a_{n-1} \equiv a_n) \end{aligned} \tag{3.12}$$

$$a_1 \overset{\gamma}{\equiv} a_n \le a_1 \to_i a_n, \qquad i = 1, 2, 3, 5 \tag{3.13}$$

$$a_1 \overset{\delta}{\equiv} a_n \le a_1 \to_i a_n, \qquad i = 1, 2, 4, 5 \tag{3.14}$$

$$(a_1 \overset{\gamma}{\equiv} a_n) \cap (a_1 \cup a_2 \cdots \cup a_n) = a_1 \cap a_2 \cdots \cap a_n \tag{3.15}$$

$$(a_1 \overset{\delta}{\equiv} a_n) \cap (a_1' \cup a_2' \cdots \cup a_n') = a_1' \cap a_2' \cdots \cap a_n' \tag{3.16}$$

Proof. Lattice O6 violates all of the above equations as well as n-Go. Thus for the proof we can presuppose that any OL in which they hold is an OML.

Equation (3.10) follows from definitions, replacing variables with their orthocomplements in n-Go.

Assuming (3.11), we use $a \equiv b = (a \to_1 b) \cap (b \to_1 a)$ to obtain the equivalent equation $a_1 \overset{\gamma}{\equiv} a_n = a_1 \overset{\gamma}{\equiv} a_n \cap a_n \overset{\gamma}{\equiv} a_1$, so $a_1 \overset{\gamma}{\equiv} a_n \le a_n \overset{\gamma}{\equiv} a_1$. By renaming variables, the other direction of the inequality also holds, establishing n-Go. Conversely, n-Go immediately implies $a_1 \stackrel{\gamma}{\equiv} a_n = a_1 \stackrel{\gamma}{\equiv} a_n \cap a_n \stackrel{\gamma}{\equiv} a_1$. The proof for (3.12) is similar.

For (3.13) and (3.14), we demonstrate only (3.13), $i = 3$. From (3.13), by rearranging factors on the left-hand-side we have $a_1 \stackrel{\gamma}{\equiv} a_n \le a_2 \to_3 a_1$, so $a_1 \stackrel{\gamma}{\equiv} a_n \le (a_2 \to_3 a_1) \cap (a_1 \to_1 a_2) = a_1 \equiv a_2$ (from Table 1 in ref. 5), etc.; in this way we build up (3.11). For the converse, (3.13) and (3.14) obviously follow from (3.11) and (3.12).

For (3.15), using (3.9), we can write (3.11) as $a_1 \stackrel{\gamma}{\equiv} a_n = (a_1 \cdots \cap a_n) \cup (a_1' \cdots \cap a_n')$. Multiplying both sides by $a_1 \cdots \cup a_n$ and using FH, we obtain (3.15). Conversely, disjoining both sides of (3.15) with $a_1' \cdots \cap a_n'$ and using FH and (3.9), we obtain (3.11). The proof for (3.16) is similar. ∎

Theorem 3.13. In any nGO, $n = 3, 4, 5, \ldots$, all of the following relations hold:

$$a_1 \stackrel{\gamma}{\equiv} a_n \le a_j \to_i a_k, \qquad 0 \le i \le 5, \quad 1 \le j \le n, \quad 1 \le k \le n \tag{3.17}$$

$$a_1 \stackrel{\delta}{\equiv} a_n \le a_j \to_i a_k, \qquad 0 \le i \le 5, \quad 1 \le j \le n, \quad 1 \le k \le n \tag{3.18}$$

$$a_1 \stackrel{\gamma}{\equiv} a_n = a_1 \stackrel{\delta}{\equiv} a_n \tag{3.19}$$

Proof. These obviously follow from (3.11) and (3.12) and (for $i = 0$) the fact that $a \to_m b \le a \to_0 b$, $0 \le m \le 5$. ∎

Some of the equations of Theorem 3.13 (in addition to those mentioned in Theorem 3.12) also imply the nGO laws. In Theorem 3.15 below we show them for $n = 3$. First we prove the following preliminary results.

Lemma 3.14. The following equations hold in all OMLs:

$$(a \to_2 b) \cap (b \to_1 c) = (a' \cap b') \cup (b \cap c) \tag{3.20}$$

$$(a_1 \to_5 a_2) \cap (a_2 \to_5 a_3) \cap (a_3 \to_5 a_1) = (a_1 \equiv a_2) \cap (a_2 \equiv a_3) \tag{3.21}$$

Proof. For (3.20), $(a \to_2 b) \cap (b \to_1 c) = ((b \cup (a' \cap b')) \cap (b' \cup (b \cap c)) = ((b \cup (a' \cap b')) \cap b') \cup ((b \cup (a' \cap b')) \cap b \cap c) = (a' \cap b') \cup (b \cap c)$.

For (3.21), we have

$$\begin{aligned}
&(a_1 \to_5 a_2) \cap (a_2 \to_5 a_3) \cap (a_3 \to_5 a_1)\\
&\quad = [(a_1 \equiv a_2) \cup (a_1' \cap a_2)] \cap [(a_2 \to_1 a_3) \cap (a_2 \to_2 a_3)]\\
&\qquad \cap [(a_3 \to_1 a_1) \cap (a_3 \to_2 a_1)]\\
&\quad \le ((a_1 \equiv a_2) \cup (a_1' \cap a_2)) \cap (a_2 \to_2 a_3) \cap (a_3 \to_1 a_1)
\end{aligned}$$

$$= ((a_1 \equiv a_2) \cup (a_1' \cap a_2)) \cap ((a_2' \cap a_3') \cup (a_3 \cap a_1))$$

$$= ((a_1 \equiv a_2) \cap ((a_2' \cap a_3') \cup (a_3 \cap a_1)))$$

$$\cup ((a_1' \cap a_2) \cap ((a_2' \cap a_3') \cup (a_3 \cap a_1)))$$

$$= ((a_1 \equiv a_2) \cap ((a_2' \cap a_3') \cup (a_3 \cap a_1))) \cup 0$$

$$\le a_1 \equiv a_2$$

In the third step, we used (3.20); in the fourth, $a_1 \equiv a_2 C a_1' \cap a_2$ and $a_1' \cap a_2 C (a_2' \cap a_3') \cup (a_3 \cap a_1)$; in the fifth, $a_1' \cap a_2 \, C a_2' \cap a_3'$ and $a_1' \cap a_2 \, C a_3 \cap a_1$. Rearranging the left-hand side, this proof also gives us $(a_1 \to_5 a_2) \cap (a_2 \to_5 a_3) \cap (a_3 \to_5 a_1) \le a_2 \equiv a_3$ and thus $\le (a_1 \equiv a_2) \cap (a_2 \equiv a_3)$. The other direction of the inequality follows from $a \equiv b \le a \to_5 b$. ■

Theorem 3.15. When $n = 3$, an OML in which any of the following holds is an nGO and vice versa:

$$a_1 \overset{\gamma}{\equiv} a_n \le a_n \to_i a_1, \qquad i = 2, 3, 4, 5 \tag{3.22}$$

$$a_1 \overset{\delta}{\equiv} a_n \le a_n \to_i a_1, \qquad i = 1, 3, 4, 5 \tag{3.23}$$

$$a_1 \overset{\gamma}{\equiv} a_n = a_1 \overset{\delta}{\equiv} a_n \tag{3.24}$$

Proof. We have already proved the converses in Theorem 3.13.

From (3.22), we have $a_1 \overset{\gamma}{\equiv} a_3 \le (a_3 \to_i a_1) \cap (a_3 \to_1 a_1) = a_3 \to_5 a_1$ since $(a \to_j b) \cap (a \to_k b) = a \to_5 b$ when $j \ne k$ for $j, k = 1, \ldots, 5$. By rearranging the left-hand side we also have $a_1 \overset{\gamma}{\equiv} a_3 \le a_1 \to_5 a_2$ and $\le a_2 \to_5 a_3$. Thus $a_1 \overset{\gamma}{\equiv} a_3 \le (a_1 \to_5 a_2) \cap (a_2 \to_5 a_3) \cap (a_3 \to_5 a_1) \le a_1 \equiv a_2 \le a_2 \to_1 a_1$, which is the 3GO law by (3.13). In the penultimate step we used (3.21).

The proof for (3.23) is similar, and from (3.24) we obtain (3.22). ■

Whether Theorem 3.15 holds for $n > 3$ is not known.

The equations obtained by substituting $\to_2$ for one or more $\to_1$'s in Godowski's equations also hold in some nGO, although to show such an equation with j variables may require the use of an n-Go equation with $n > j$.

Theorem 3.16. The following equation with i variables holds in some nGO with $n \ge i$, where each $\to_{j_k}$ $(1 \le k \le i)$ is either $\to_1$ or $\to_2$ in any combination:

$$(a_1 \to_{j_1} a_2) \cap (a_2 \to_{j_2} a_3) \cdots \cap (a_{i-1} \to_{j_{i-1}} a_i)$$

$$\cap\ (a_i \to_{j_i} a_1) = a_1 \overset{\gamma}{\equiv} a_i \tag{3.25}$$

Proof. We illustrate the proof by showing that the three-variable equation

$$(a_1 \to_2 a_2) \cap (a_2 \to_1 a_3) \cap (a_3 \to_1 a_1) = a_1 \overset{\gamma}{\equiv} a_3 \tag{3.26}$$

holds in any 4GO. The essential identities we use are

$$(a \to_1 (a \cup b)) \cap ((a \cup b) \to_1 b) = a \to_2 b \tag{3.27}$$

$$(a \to_2 (a \cap b)) \cap ((a \cap b) \to_2 b) = a \to_1 b \tag{3.28}$$

which hold in any OML. Starting with (3.17), we have

$$\begin{aligned}
&(a_1 \to_1 b) \cap (b \to_1 a_2) \cap (a_2 \to_1 a_3) \cap (a_3 \to_1 a_1)\\
&\quad \le (a_1 \to_1 a_3) \cap (a_3 \to_1 a_1) \cap (a_3 \to_1 a_2) \cap (a_2 \to_1 a_3)\\
&\quad = (a_1 \equiv a_3) \cap (a_3 \equiv a_2)\\
&\quad = (a_1 \equiv a_2) \cap (a_2 \equiv a_3)\\
&\quad = a_1 \overset{\gamma}{\equiv} a_3
\end{aligned}$$

where in the penultimate step we used (2.9) [or more generally (3.9)] and in the last step (3.11). Substituting $a_1 \cup a_2$ for b and using (3.27), we obtain

$$(a_1 \to_2 a_2) \cap (a_2 \to_1 a_3) \cap (a_3 \to_1 a_1) \le a_1 \overset{\gamma}{\equiv} a_3$$

Using (3.17) for the other direction of the inequality, we obtain (3.26). The reader should be able to construct the general proof. ■

A consequence of (3.25) that holds in any 4GO is

$$(a \to_1 b) \cap (b \to_2 c) \cap (c \to_1 a) \le (a \equiv c) \tag{3.29}$$

which, using (3.20) and weakening the leftmost factor, implies

$$(a \equiv b) \cap ((b' \cap c') \cup (a \cap c)) \le (a \equiv c) \tag{3.30}$$

Equation (3.30) is also a consequence of (2.10), as can be seen if we write (2.10) as follows:

$$(a \equiv b) \cap ((b \cap c) \cup (b' \cap c') \cup (a \cap c) \cup (a' \cap c')) \le (a \equiv c)$$

As with (2.10), we were unable to prove that even the weaker looking (3.30) holds in all OMLs. It is also unknown if (3.30) even holds in all 3GOs. Finally, we do not know if there is an n such that (2.10) holds in all nGOs.

The relations obtained by substituting $\to_i$ for $\to_1$ in the Godowski equations do not in general result in equivalents for $i = 3, 4, 5$, nor even hold in an nGO. For example, for $n = 3$, the conditions $(a_1 \to_3 a_2) \cap (a_2 \to_3 a_3) \cap (a_3 \to_3 a_1) \le (a_2 \to_3 a_1)$ and $(a_1 \to_4 a_2) \cap (a_2 \to_4 a_3) \cap (a_3 \to_4 a_1) \le (a_2 \to_4 a_1)$ fail in lattice MO2 (*Chinese lantern*, Fig. 1b), and $(a_1 \to_5 a_2) \cap (a_2 \to_5 a_3) \cap (a_3 \to_5 a_1) \le (a_2 \to_5 a_1)$ holds in all OMLs by (3.21).

Lemma 3.17. Any nGO is an $(n - 1)$GO, $n = 4, 5, 6, \ldots$.

Proof. Substitute a_1 for a_2 in equation n-Go. ■

The converse of Lemma 3.17 does not hold. Indeed, the *wagon wheel* OMLs Gn, $n = 3, 4, 5, \ldots$, are related to the n-Go equations in the sense that Gn violates n-Go, but (for $n \ge 4$) not $(n - 1)$-Go. In Fig. 2 we show examples G3 and G4; for larger n we construct Gn by adding more "spokes" in the obvious way (according to the general scheme described in ref. 25).

For any particular n there may exist lattices smaller than Gn for which this property holds. These can be more efficient, computationally, for proving that an equation derived in nGO is weaker than n-Go or independent of $(n - 1)$-Go. Based on a computer scan of all (legless) OMLs with 3-atom blocks (see footnote at the end of Section 2), up to and including a block count of 12 along with selected lattices with block counts up to 17, we obtained the following results. Lattice G3, with 34 nodes, is the smallest that violates 3-Go. (In OMLs with 3-atom blocks, the number of nodes is twice the number of atoms, plus 2.) The Peterson OML, with 32 nodes (vs. 44 nodes in G4), is the smallest that violates 4-Go, but not 3-Go (Fig. 3). Lattice G5s, with 42 nodes (vs. 54 nodes in G5), is the smallest that violates 5-Go, but not 4-Go (also Fig. 3). Lattices G6s1 and G6s2, each with 44 nodes (vs. 64 nodes in G6), are two of three smallest that violate 6-Go, but not 5-Go (Fig. 4), Of these three, G6sl is one of two with 14 blocks, whereas G6s2 has 15 blocks. Lattices G7sl and G7s2 (Fig. 5) are two of several smallest

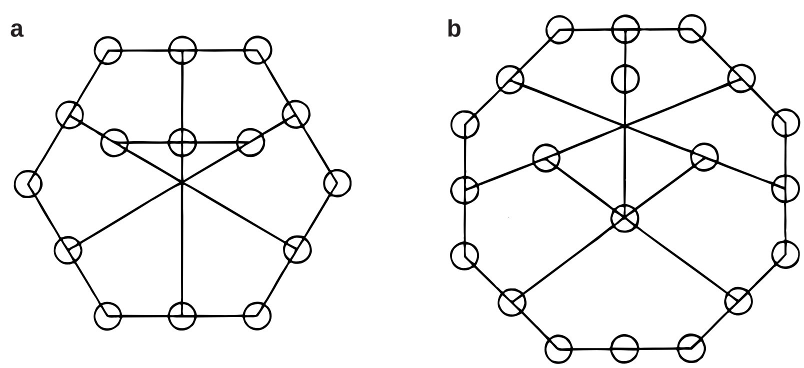

Fig. 3. (a) Peterson OML. (b) Greechie diagram for OML G5s.

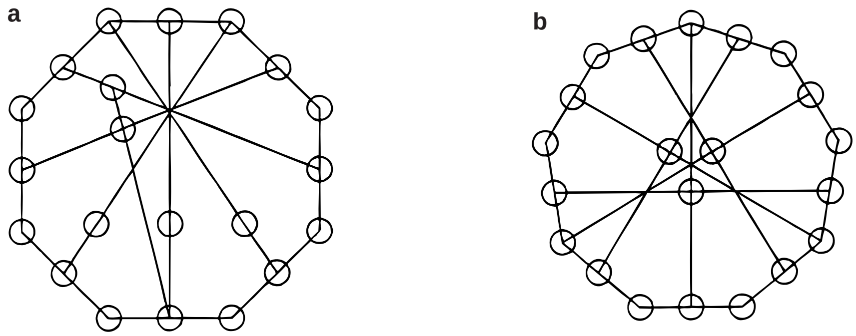

Fig. 4. Greechie diagrams for (a) OML G6s1 and (b) OML G6s2.

we obtained to violate 7-Go, but not 6-Go. They both have 50 nodes and 16 and 17 blocks, respectively (vs. 74 nodes and 21 blocks in G7). Actually, we used dynamic programming to obtain a program for checking on n-Go which is so fast that no reasonable n is a problem. For example, to find G7sl among 207,767 Greechie diagrams with 24 atoms and 16 blocks took an 800 MHz PC less than 2 hr.

The next lemma provides some technical results for subsequent use. Note that $a \perp b \perp c$ means $a \perp b$ and $b \perp c$ (but not necessarily $a \perp c$).

Lemma 3.18. In any OML we have

$$a \perp b \perp c \Rightarrow (a \cup b) \cap (a \to_2 c) \le b \cup c \tag{3.31}$$

$$a \perp b \perp c \Rightarrow a \cup b \le c \to_2 a \tag{3.32}$$

$$a \perp b \perp c \quad \& \quad (c \to_2 a) \cap d \le a \to_2 c \Rightarrow (a \cup b) \cap d \le b \cup c \tag{3.33}$$

Proof. For (3.31), $(a \cup b) \cap (a \to_2 c) = (a \cup b) \cap (c \cup (a' \cap c'))$. From hypotheses, b commutes with a and $c \cup (a' \cap c')$. Using FH twice, $(a \cup b) \cap (c \cup (a' \cap c')) = (b \cap (c \cup (a' \cap c'))) \cup (a \cap c) \cup (a \cap a' \cap$

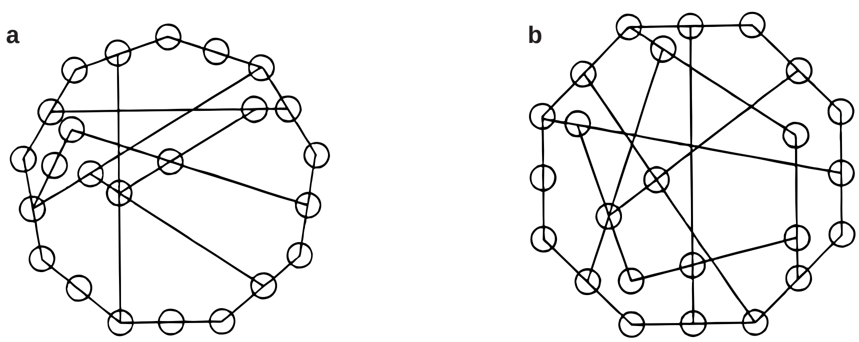

Fig. 5. Greechie diagrams for (a) OML G7s1 and (b) OML G7s2.

$c') \le b \cup c$. For (3.32), from hypotheses, $a \cup b \le a \cup (c' \cap a') = c \to_2 a$. For (3.33), from (3.32), $(a \cup b) \cap d \le (c \to_2 a) \cap d \le$ [from hypothesis] $a \to_2 c$. Thus $(a \cup b) \cap d \le (a \cup b) \cap (a \to_2 c)$, which by (3.31) is $\le b \cup c$. ■

The n-Go equations can be equivalently expressed as inferences involving $2n$ variables, as the following theorem shows. In this form they can be useful for certain kinds of proofs, as we illustrate in Theorem 3.20.

Theorem 3.19. Any OML in which

$$a_1 \perp b_1 \perp a_2 \perp b_2 \perp \dots a_n \perp b_n \perp a_1$$
$$\Rightarrow (a_1 \cup b_1) \cap (a_2 \cup b_2) \cap \cdots (a_n \cup b_n) \le b_1 \cup a_2 \qquad (3.34)$$

holds is an nGO and vice versa.

Proof. Substituting c_1 for $a_1, \dots, c_n$ for a_n; $c'_1 \cap c'_2$ for $b_1, \dots, c'_{n-1} \cap c'_n$ for b_{n-1}; and $c'_n \cap c'_1$ for b_n; we satisfy the hypotheses of (3.34) and obtain (3.14).

Conversely, suppose the hypotheses of (3.34) hold. From the hypotheses and (3.32), we obtain $(a_2 \cup b_2) \cdots \cap (a_{n-1} \cup b_{n-1}) \cap (a_n \cup b_n) \le (a_3 \to_2 a_2) \cdots \cap (a_n \to_2 a_{n-1}) \cap (a_1 \to_2 a_n)$. Thus $(a_2 \to_2 a_1) \cap [(a_2 \cup b_2) \cdots \cap (a_{n-1} \cup b_{n-1}) \cap (a_n \cup b_n)] \le (a_2 \to_2 a_1) \cap [(a_3 \to_2 a_2) \cdots \cap (a_n \to_2 a_{n-1}) \cap (a_1 \to_2 a_n)] = (a_2 \to_2 a_1) \cap (a_1 \to_2 a_n) \cap (a_n \to_2 a_{n-1}) \cdots \cap (a_3 \to_2 a_2)$. Applying (3.14) to the right-hand side, we obtain $(a_2 \to_2 a_1) \cap [(a_2 \cup b_2) \cdots \cap (a_{n-1} \cup b_{n-1}) \cap (a_n \cup b_n)] \le a_1 \to_2 a_2$. Then (3.33) gives us (3.34). ■

Mayet [26] presents a method for obtaining equations that hold in all lattices with a strong or full set of states. However, it turns out that the examples of those equations he shows are implied by the n-Go equations and thus do not provide us with additional information about lattices with strong states or $\mathcal{C}(\mathcal{H})$ in particular. To the authors' knowledge, there is no known example of such an equation that cannot be derived from the n-Go equations. It apparently remains an open problem whether Mayet's method gives equations that hold in all OMLs with a strong set of states, but that cannot be derived from equations n-Go.

Theorem 3.20. The following conditions (derived as Examples 2–4 in ref. 26) hold in 3GO, 6GO, and 4GO, respectively:

$$(a \to_1 b) \cap (b \to_1 c) \cap (c \to_1 a) \le b \to_1 a \qquad (3.35)$$

$$a \perp b \perp c \perp d \perp e \perp f \perp a$$
$$\Rightarrow (a \cup b) \cap (d \cup e)' \cap ((((a \cup b) \to_1 (d \cup e)') \to_1 ((e \cup f)$$
$$\to_1 (b \cup c)')')' \to_1 (c \cup d)) \le b \cup c \cup (e \cup f)' \qquad (3.36)$$

$$a \perp b \perp c \perp d \perp e \perp f \perp g \perp h \perp a$$
$$\Rightarrow (a \cup b) \cap (c \cup d) \cap (e \cup f) \cap (g \cup h)$$
$$\cap ((a \cup h) \rightarrow_1 (d \cup e)') = 0 \tag{3.37}$$

Proof. For (3.35), this is the same as (3.13) for $n = 3$.

For (3.36), using (3.34), we express the 6GO law as

$$a_1 \perp b_1 \perp a_2 \perp b_2 \perp a_3 \perp b_3 \perp a_4 \perp b_4 \perp a_5 \perp b_5 \perp a_6 \perp b_6 \perp a_1$$
$$\Rightarrow (a_1 \cup b_1) \cap (a_2 \cup b_2) \cap (a_3 \cup b_3) \cap (a_4 \cup b_4) \cap (a_5 \cup b_5)$$
$$\cap (a_6 \cup b_6) \leq b_1 \cup a_2 \tag{3.38}$$

We define $p = ((a \cup b) \rightarrow_1 (d \cup e)')'$, $q = ((e \cup f) \rightarrow_1 (b \cup c)')'$, and $r = (p' \rightarrow_1 q)' \cap (c \cup d)$. In (3.38) we substitute a for a_1, b for b_1, c for a_2, $(c \cup d)'$ for b_2, r for a_3, $p' \rightarrow_1 q$ for b_3, $(p' \rightarrow_1 q)'$ for a_4, $p' \cap q$ for b_4, q' for a_5, q for b_5, $(e \cup f)'$ for a_6, and f for b_6. With these substitutions, all hypotheses of (3.38) are satisfied by the hypotheses of (3.36). The conclusion becomes

$$(a \cup b) \cap (c \cup (c \cup d)') \cap (r \cup (p' \rightarrow_1 q))$$
$$\cap ((p' \rightarrow_1 q)' \cup (p' \cap q)) \cap (q' \cup q) \cap ((e \cup f)' \cup f)$$
$$\leq b \cup c \tag{3.39}$$

We simplify (3.39) using $c \cup (c \cup d)' =$ [since c and d commute by hypothesis] $(c \cup c') \cap (c \cup d') = 1 \cap (c \cup d') =$ [since $c \leq d'$] d'; $(e \cup f)' \cup f = e'$ similarly; $(p' \rightarrow_1 q)' \cup (p' \cap q) = p'$; and $q' \cup q = 1$. This gives us

$$(a \cup b) \cap d' \cap (r \cup (p' \rightarrow_1 q)) \cap p' \cap e' \leq b \cup c \tag{3.40}$$

Now, in any OML we have $p' = (a \cup b) \rightarrow_1 (d \cup e)' = (a \cup b)' \cup ((a \cup b) \cap (d \cup e)') \geq (a \cup b) \cap (d \cup e)' \geq (a \cup b) \cap (d \cup e)' \cap ((p' \rightarrow_1 q) \cup r)$. Thus the left-hand side of (3.40) absorbs p', so

$$(a \cup b) \cap d' \cap (r \cup (p' \rightarrow_1 q)) \cap e' \leq b \cup c$$
$$\leq b \cup c \cup (e \cup f)' \tag{3.41}$$

which after rearranging is exactly (3.36).

For (3.37), using (3.34), we obtain from the 4GO law

$$a \perp b \perp c \perp d \perp e \perp f \perp g \perp h \perp a$$
$$\Rightarrow (a \cup b) \cap (c \cup d) \cap (e \cup f) \cap (g \cup h) \leq (a \cup h) \cap (d \cup e)$$

Therefore,

$$
\begin{aligned}
&a \perp b \perp c \perp d \perp e \perp f \perp g \perp h \perp a \\
&\quad \Rightarrow (a \cup b) \cap (c \cup d) \cap (e \cup f) \cap (g \cup h) \cap ((a \to h) \to_1 (d \cup e)') \\
&\qquad \le (a \cup h) \cap (d \cup e) \cap ((a \cup h) \to_1 (d \cup e)')
\end{aligned}
$$

In any OML we have $x \cap y \cap (x \to_1 y') = 0$; applying this to the right-hand side, we obtain (3.37). ■

To the authors' knowledge all 3-variable conditions published so far that hold in all OMLs with a strong set of states are derivable in 3GO. Below we show an equation with 3 variables that is derivable in 6GO, but is independent of the 3GO law. It shows that it is possible to express with only 3 variables a property that holds only in nGOs smaller than 3GO.

Theorem. 3.21. The 3-variable condition

$$
\begin{aligned}
&((a \to_2 b) \cap (a \to_2 c)') \cap ((((a \to_2 b) \to_1 (a \to_2 c)') \\
&\quad \to_1 ((b \to_2 c) \to_1 (b \to_2 a)')')' \\
&\quad \to_1 (c \to_2 a)) \le b \to_2 a
\end{aligned} \tag{3.42}
$$

holds in a 6GO, but cannot be derived (in an OML) from the 3GO law nor vice versa.

Proof. To show this equation holds in 6GO, we start with (3.41), which occurs in the proof of Mayet's Example 3, rewriting it as

$$
\begin{aligned}
&d \perp e \perp f \perp g \perp h \perp j \perp d \\
&\quad \Rightarrow (d \cup e) \cap (g \cup h)' \cap ((((d \cup e) \to_1 (g \cup h)') \to_1 ((h \cup j) \\
&\qquad \to_1 (e \cup f)')')' \to_1 (f \cup g)) \le e \cup f
\end{aligned} \tag{3.43}
$$

We substitute b for d, $a' \cap b'$ for e, a for f, $a' \cap c'$ for g, c for h, and $c' \cap b'$ for j. With these substitutions, the hypotheses of (3.43) are satisfied. This results in (3.42), showing that (3.42) holds in 6GO.

We show independence as follows. On the one hand, (3.42) fails in the Peterson OML (Fig. 3a), but holds in OML G3 (Fig. 2a). On the other hand, the 3GO law (3.6) holds in the Peterson OML, but fails in G3. ■

It is not known whether (3.42) holds in 4GO or 5GO.

Using our results so far, we can show that $a_i \overset{\gamma}{\equiv} a_j = 1$ is similar to a relation of equivalence (although strictly speaking it is not one, since $a_i \overset{\gamma}{\equiv} a_j$ involves not 2, but $|j - i| + 1$ variables). Reflexivity $a_i \overset{\gamma}{\equiv} a_i = 1$ follows by definition, symmetry $a_i \overset{\gamma}{\equiv} a_j = 1 \Rightarrow a_j \overset{\gamma}{\equiv} a_i = 1$ from the Godowski equations, and transitivity $a_i \overset{\gamma}{\equiv} a_j = 1$ & $a_j \overset{\gamma}{\equiv} a_k = 1 \Rightarrow a_i \overset{\gamma}{\equiv} a_k = 1$ from the following theorem. Analogous results can be stated for $\overset{\delta}{\equiv}$.

An open problem is whether there exists an equation corresponding to $a_i \overset{\gamma}{\equiv} a_j = 1$ and $a_i \overset{\delta}{\equiv} a_j = 1$ as in (2.3) and (4.4)–(4.6).

Theorem 3.22. The following holds in nGO, where $i, j \geq 1$ and $n = \max(i, j, 3)$:

$$(a_1 \overset{\gamma}{\equiv} a_i) \cap (a_i \overset{\gamma}{\equiv} a_j) \leq a_1 \overset{\gamma}{\equiv} a_j \tag{3.44}$$

Proof. If $\overset{\gamma}{\equiv}$ has 3 or more variables, we replace it with a chained identity per (3.11), otherwise we replace it with the extended definition we mention after Definition 3.9. The proof is then obvious. (In many cases the equation may also hold for smaller n or even in OML or OL, e.g., when $j = 1$.) ■

The next lemma shows an interesting "variable-swapping" property of the Godowski identity that we shall use in a later proof [of Theorem 6.6].

Lemma 3.23. In any OML we have

$$(a_1 \overset{\gamma}{\equiv} a_n) \cap a_i' = a_1' \cap a_2' \cdots \cap a_n', \qquad i = 1, \ldots, n \tag{3.45}$$

In particular,

$$(a_1 \overset{\gamma}{\equiv} a_n) \cap a_i' = (a_1 \overset{\gamma}{\equiv} a_n) \cap a_j', \qquad i = 1, \ldots, n, \quad j = 1, \ldots, n \tag{3.46}$$

Proof. We illustrate the case $i = 1$. In any OML we have $a' \cap (b \rightarrow_1 a) = a' \cap b'$. Thus $(a_1 \rightarrow_1 a_2) \cdots \cap (a_{n-2} \rightarrow_1 a_{n-1}) \cap (a_{n-1} \rightarrow_1 a_n) \cap (a_n \rightarrow_1 a_1) \cap a_1' = (a_1 \rightarrow_1 a_2) \cdots \cap (a_{n-2} \rightarrow_1 a_{n-1}) \cap (a_{n-1} \rightarrow_1 a_n) \cap a_n' \cap a_1' = \cdots = a_1' \cap a_2' \cdots \cap a_{n-1}' \cap a_n' \cap a_1'$. ■

4. ORTHOARGUESIAN EQUATIONS

In this section we show that all orthoarguesian-based equalities (which must hold in any Hilbert lattice) that have appeared in the literature as equations with 4 and 6 variables can be reduced to just two equations with 3 and 4 variables. The latter two equations we call the 3OA and 4OA laws, respectively, and introduce them by Definition 4.4. Their equivalence to the afore mentioned 4- and 6-variable equations is shown in Theorems 4.8 and 4.9 and in Theorem 4.7, respectively. A new 3-variable consequence of the 4OA law which is not equivalent to the 3OA law is given by Theorem 4.11. Possibly equivalent inference forms of the 3OA law and the 4OA law are given by Theorems 4.2, 4.3, and 4.10.

Definition 4.1. We have

$$a \overset{c}{\equiv}_i b \overset{\text{def}}{=} ((a \to_i c) \cap (b \to_i c)) \cup ((a' \to_i c) \cap (b' \to_i c)),\quad i = 1, 3 \tag{4.1}$$

$$a \overset{c}{\equiv}_i b \overset{\text{def}}{=} ((c \to_i a) \cap (c \to_i b)) \cup ((c \to_i a') \cap (c \to_i b')),\quad i = 2, 4 \tag{4.2}$$

$$a \overset{c,d}{\equiv}_i b \overset{\text{def}}{=} (a \overset{d}{\equiv}_i b) \cup ((a \overset{d}{\equiv}_i c) \cap (b \overset{d}{\equiv}_i c))\quad i = 1, \dots, 4 \tag{4.3}$$

We call $a \overset{c}{\equiv}_i b$ a 3-variable orthoarguesian identity and $a \overset{c,d}{\equiv}_i b$ a 4-variable orthoarguesian identity and denote them as 3-oa and 4-oa, respectively.

Theorem 4.2. An ortholattice to which any of

$$a \overset{c}{\equiv}_i b = 1 \Leftrightarrow a \to_i c = b \to_i c, \qquad i = 1, 3 \tag{4.4}$$

$$a \overset{c}{\equiv}_i b = 1 \Leftrightarrow c \to_i a = c \to_i b, \qquad i = 2, 4 \tag{4.5}$$

is added is a variety smaller than OML that fails in lattice L28 (Fig. 6a).

The corresponding expressions for $i = 5$ do not hold in a Hilbert lattice (right to left implications fail in MO2).

Theorem 4.3. An ortholattice to which any of

$$a \overset{c,d}{\equiv}_i b = 1 \Leftrightarrow a \to_i d = b \to_i d, \qquad i = 1, 3 \tag{4.6}$$

is added is a variety smaller than OML that fails in lattice L36 (Fig. 6b) for $i = 1, 3$.

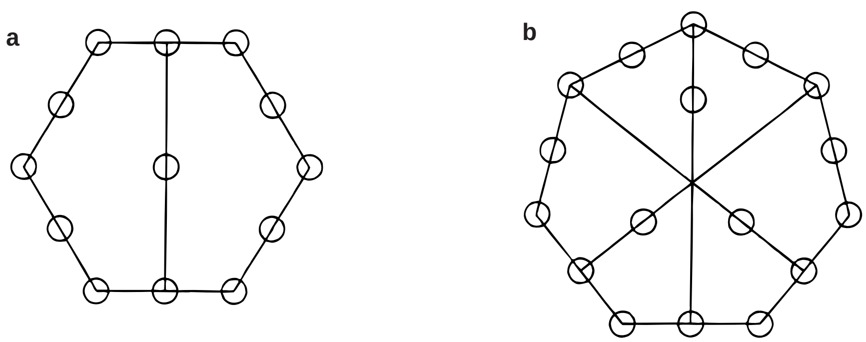

Fig. 6. Greechie diagrams for (a) OML L28 and (b) OML L36.

The new identities $\stackrel{c}{\equiv}_1$ and $\stackrel{c,d}{\equiv}_1$ being equal to one, are relations of equivalence. It is obvious that they are reflexive ($a \stackrel{c}{\equiv}_1 a = 1$, $a \stackrel{c,d}{\equiv}_1 a = 1$) and symmetric ($a \stackrel{c}{\equiv}_1 b = 1 \Rightarrow b \stackrel{c}{\equiv}_1 a = 1$, $a \stackrel{c,d}{\equiv}_1 b = 1 \Rightarrow b \stackrel{c,d}{\equiv}_1 a = 1$), and the transitivity follows from Theorem 4.10 below. They are, however, not relations of congruence because $a \stackrel{c}{\equiv}_1 b = 1 \Rightarrow (a \cup d) \stackrel{c}{\equiv}_1 (b \cup d) = 1$ does not hold: it fails in the Chinese lantern MO2 (Fig. 1b). Conditions (4.4)–(4.6) must hold in any Hilbert space (and therefore by any quantum simulator) for $i = 1$ as we show below. Expressions corresponding to (4.6) for $i = 2, 4, 5$ do not hold in a Hilbert lattice and it is an open problem whether there exist equivalent relations of equivalence for $i = 2, 4, 5$. In what follows we keep to $i = 1$ (and not $i = 3$) because $i = 1$ enables us to switch to the Sasaki projection $\varphi_a b = (a \rightarrow_1 b')'$ of b on a later. The Sasaki projection plays an important role in the definition of the covering property, which is a consequence of the superposition principle [19].

Definition 4.4. Let $a \stackrel{c}{\equiv} b \stackrel{\text{def}}{=} a \stackrel{c}{\equiv}_1 b$ and $a \stackrel{c,d}{\equiv} b \stackrel{\text{def}}{=} a \stackrel{c,d}{\equiv}_1 b$.

A 3OA is an OL in which the following additional condition is satisfied:

$$(a \rightarrow_1 c) \cap (a \stackrel{c}{\equiv} b) \leq b \rightarrow_1 c \tag{4.7}$$

A 4OA is an OL in which the following additional condition is satisfied:

$$(a \rightarrow_1 d) \cap (a \stackrel{c,d}{\equiv} b) \leq b \rightarrow_1 d \tag{4.8}$$

Note that the 3OA and 4OA laws (4.7) and (4.8) have three and four variables, respectively. Both 3OA and 4OA laws, fail in O6, so they are OMLs, but there exist OMLs that are neither 3OAs nor 4OAs: conditions (4.7) and (4.8) both fail in the orthomodular lattice L28 (Fig. 6a).

Theorem 4.5. Every 4OA is a 3OA, but there exist 3OAs that are not 4OAs.

Proof. In $(a \rightarrow_1 d) \cap (a \stackrel{c,d}{\equiv} b) \leq (b \rightarrow_1 d)$, set $c = b$. On the other hand, lattice L36 (Fig. 6b) is a 3OA because it is an OML in which (4.7) holds, but it is not a 4OA because it violates (4.8). ■

The next lemma provides some technical results for use in subsequent proofs.

Lemma 4.6. In any OML we have

$$(a \rightarrow_1 b) \cap a = a \cap b \tag{4.9}$$

$$(a \rightarrow_1 b) \cap (a' \rightarrow_1 b) = (a \rightarrow_1 b) \cap b = (a \cap b) \cup (a' \cap b) \tag{4.10}$$

$$(a' \to_1 b)' \le a' \le a \to_1 b \tag{4.11}$$

$$(a \to_1 b) \to_1 b = a' \to_1 b \tag{4.12}$$

$$(a \to_1 b)' \to_1 b = a \to_1 b \tag{4.13}$$

$$(a \to_i b) \cup (a \to_j b) = a \to_0 b, \qquad i, j = 0, \ldots, 4, \quad i \ne j \tag{4.14}$$

$$a' \le b \Rightarrow b \le a \to_1 b \tag{4.15}$$

$$a \cap ((a \to_1 c) \cup b) \le c \Leftrightarrow b \le a \to_1 c \tag{4.16}$$

Proof. For (4.9)–(4.15), we omit the easy proofs.

For (4.16), if $a \cap ((a \to_1 c) \cup b) \le c$, then $a \cap ((a \to_1 c) \cup b) \le a \cap c = (a \to_1 c) \cap a$ using (4.9), so $b \le 1 \cap ((a \to_1 c) \cup b) = ((a \to_1 c) \cup a) \cap ((a \to_1 c) \cup ((a \to_1 c) \cup b)) =$ (via FH) $(a \to_1 c) \cup (a \cap ((a \to_1 c) \cup b)) \le (a \to_1 c) \cup ((a \to_1 c) \cap a) = a \to_1 c$. Conversely, if $b \le a \to_1 c$, then, using (4.9), $a \cap ((a \to_1 c) \cup b) = a \cap (a \to_1 c) = a \cap c \le c$. ∎

In the next theorem we show that the 4OA law (4.8) is equivalent to the orthoarguesian law (4.17) discovered by Day [27, 28], which holds in $\mathcal{C}(\mathcal{H})$. Thus the 4OA law also holds in $\mathcal{C}(\mathcal{H})$.

Theorem 4.7. An OML in which

$$\begin{aligned} & a \perp b \;\&\; c \perp d \;\&\; e \perp f \\ & \quad \Rightarrow (a \cup b) \cap (c \cup d) \cap (e \cup f) \\ & \qquad \le b \cup (a \cap (c \cup (((a \cup c) \cap (b \cup d)) \cap (((a \cup e) \cap (b \cup f)) \\ & \qquad \cup ((c \cup e) \cap (d \cup f)))))) \end{aligned} \tag{4.17}$$

(where $a \perp b \overset{\text{def}}{=} a \le b'$) holds is a 4OA and vice versa.

Proof. We will work with the dual of (4.17),

$$\begin{aligned} & a' \le b \;\&\; c' \le d \;\&\; e' \le f \\ & \quad \Rightarrow b \cap (a \cup (c \cap (((a \cap c) \cup (b \cap d)) \cup (((a \cap e) \cup (b \cap f)) \\ & \qquad \cap ((c \cap e) \cup (d \cap f)))))) \\ & \qquad \le (a \cap b) \cup (c \cap d) \cup (e \cap f) \end{aligned} \tag{4.18}$$

First we show that the 4OA law implies (4.18). In any OL we have

$$\begin{aligned} & b \le g \to_1 k \;\&\; d \le h \to_1 k \;\&\; f \le j \to_1 k \\ & \quad \Rightarrow b \cap (a \cup (h \cap (((g \cap h) \cup (b \cap d)) \cup (((g \cap j) \cup (b \cap f)) \\ & \qquad \cap ((h \cap j) \cup (d \cap f)))))) \end{aligned}$$

$$\le (g \to_1 k) \cap (g \cup (h \cap (((g \cap h) \cup ((g \to_1 k) \cap h \to_1 k)))$$
$$\cup (((g \cap j) \cup ((g \to_1 k) \cap (j \to_1 k))) \cap ((h \cap j)$$
$$\cup ((h \to_1 k) \cap j \to_1 k))))))) \tag{4.19}$$

Substituting $a' \to_1 k$ for g, $c' \to_1 k$ for h, and $e' \to_1 k$ for j, simplifying with (4.12), and applying (4.11) to the left-hand side of the conclusion, we obtain

$$b \le a \to_1 k \ \& \ d \le c \to_1 k \ \& \ f \le e \to_1 k$$
$$\Rightarrow b \cap (a \cup (c \cap (((a \cap c) \cup (b \cap d)) \cup (((a \cap e) \cup (b \cap f))$$
$$\cap ((c \cap e) \cup (d \cap f))))))$$
$$\le (a \to_1 k) \cap (((a' \to_1 k) \cup ((c' \to_1 k) \cap (c \overset{e,k}{\equiv} a)))) \tag{4.20}$$

We convert the 4OA law $(c' \to_1 k) \cap (c' \overset{e,k}{\equiv} m') \le (m' \to_1 k)$ to $(c' \to_1 k) \cap (c \overset{e,k}{\equiv} m) \le (m' \to_1 k)$ to $m' \cap ((m' \to_1 k) \cup ((c' \to_1 k) \cap (c \overset{e,k}{\equiv} m))) \le k$ using (4.16). We substitute $(a \to_1 k)'$ for m and simplify with (4.12) and (4.13) to obtain $(a \to_1 k) \cap ((a' \to_1 k) \cup ((c' \to_1 k) \cap (c \overset{e,k}{\equiv} a))) \le k$. Combining with (4.20) yields

$$b \le a \to_1 k \ \& \ d \le c \to_1 k \ \& \ f \le e \to_1 k$$
$$\Rightarrow b \cap (a \cup (c \cap (((a \cap c) \cup (b \cap d)) \cup (((a \cap e) \cup (b \cap f))$$
$$\cap ((c \cap e) \cup (d \cap f)))))) \le k \tag{4.21}$$

Letting $k = (a \cap b) \cup (c \cap d) \cup (e \cap f)$, we have

$$a' \le b \Rightarrow b \le a \to_1 k \tag{4.22}$$
$$c' \le d \Rightarrow d \le c \to_1 k \tag{4.23}$$
$$e' \le f \Rightarrow f \le e \to_1 k \tag{4.24}$$

[e.g. for (4.22), using (4.15), we have $b \le a \to_1 b = a' \cup (a \cap (a \cap b)) \le a' \cup (a \cap k) = a \to_1 k$] from which we obtain (4.18).

Conversely, assume (4.18) holds. Let $a = g \to_1 k$, $b = g' \to_1 k$, $c = h \to_1 k$, $d = h' \to_1 k$, $e = j \to_1 k$, $f = j' \to_1 k$. The hypotheses of (4.18) are satisfied using (4.11). Noticing [with the help of (4.10)] that the right-hand side of the resulting inequality is $\le k$, we have $(g' \to_1 k) \cap ((g \to_1 k) \cup ((h \to_1 k) \cap (h \overset{j,k}{\equiv} g))) \le k$, so $g \cap ((g \to_1 k) \cup ((h \to_1 k) \cap (h \overset{j,k}{\equiv} g))) \le k$. Applying (4.16), we have the 4OA law $(h \to_1 k) \cap (h \overset{j,k}{\equiv} g) \le g \to_1 k$. ■

Thus we have demonstrated that the orthoarguesian law (4.17) can be expressed by an equation with only four variables instead of six. This is in contrast to the stronger Arguesian law that has been shown by Haiman to necessarily involve at least six variables [29].

The 3OA law (4.7) expresses an orthoarguesian property that does not hold in all OMLs, but as demonstrated by the fact that it holds in OML L36, it is strictly weaker than the *proper* orthoarguesian law expressed by (4.8) or (4.17). The 3OA law is equivalent to the following three-variable equation [27, Equation (III)] obtained by Godowski and Greechie and thus to the other three-variable variants of that equation mentioned in ref. 27. Godowski and Greechie were apparently the first to observe that (4.25) fails in OML L28 and also in OML $\hat{\mathrm{L}}$ of Fig. 8a below.

Theorem 4.8. An OML in which

$$\varphi_{b'}a \cup \alpha(a, b, c) = \varphi_{c'}a \cup \alpha(a, b, c) \tag{4.25}$$

[where φ is the Sasaki projection and $\alpha(a, b, c) \stackrel{\text{def}}{=} (b \cup c) \cap (\varphi_{b'}a \cup \varphi_{c'}a)$] holds is a 3OA and vice versa.

Proof. Using the definitions, (4.25) can be written in the dual form $(a \to_1 c) \cap ((a \cap b) \cup ((a \to_1 c) \cap (b \to_1 c))) = (b \to_1 c) \cap ((a \cap b) \cup ((a \to_1 c) \cap (b \to_1 c)))$. We substitute $a' \to_1 c$ for a and $b' \to_1 c$ for b throughout; simplifying with (4.12), we obtain $(a \to_1 c) \cap (a \stackrel{c}{\equiv} b) = (b \to_1 c) \cap (a \stackrel{c}{\equiv} b)$. This is easily shown to be equivalent to (4.7). ∎

Equation (4.25) was derived by Godowski and Greechie from Eq. (4.26) below, which is a four-variable substitution instance of (4.17). Godowski and Greechie state that (4.25) is “more restrictive” than (4.26). While it is not clear to us what is meant by this remark, it turns out that the two equations are equivalent in an OML. This equivalence also means that the 4OA law cannot be derived from (4.26) [which can also be verified independently by noticing that (4.26) does not fail in OML L36].

Theorem 4.9. An OML in which

$$\begin{aligned} &a \perp b \ \&\ c \perp d \\ &\quad \Rightarrow (a \cup b) \cap (c \cup d) \le b \cup (a \cap (c \cup ((a \cup c) \cap (b \cup d)))) \end{aligned} \tag{4.26}$$

holds is a 3OA and vice versa.

Proof. The proof is analogous to that for Theorem 4.7. ∎

With the help of the following theorem, we show that the relation of equivalence introduced in Theorems 4.2 and 4.3 is transitive.

Theorem 4.10. (a) In any 3OA we have

$$a \overset{c}{\equiv} b = 1 \Leftrightarrow a \to_1 c = b \to_1 c \tag{4.27}$$

(b) In any 40A we have

$$a \overset{c,d}{\equiv} b = 1 \Leftrightarrow a \to_1 d = b \to_1 d \tag{4.28}$$

Proof. For (4.27), assuming $a \overset{c}{\equiv} b = 1$, we have $(a \to_1 c) \cap (a \overset{c}{\equiv} b) = (a \to_1 c) \cap 1 \le (b \to_1 c)$ by (4.7). Conversely, from (4.11), $(a \to_1 c)' \le a' \to_1 c$ and $(b \to_1 c)' \le b' \to_1 c$, and from the hypothesis, $(a \to_1 c) \equiv (b \to_1 c) = 1$, so $1 = ((a \to_1 c) \cap (b \to_1 c)) \cup ((a \to_1 c)' \cap (b \to_1 c)') \le ((a \to_1 c) \cap (b \to_1 c)) \cup ((a' \to_1 c) \cap (b' \to_1 c)) = a \overset{c}{\equiv} b$. For (4.28), the proof is similar, noticing that, for the converse, $a \overset{d}{\equiv} b \le a \overset{c,b}{\equiv} b$. ■

The inference rules (4.27) and (4.28) fail in lattices L28 and L36, respectively, suggesting the possibility that they imply (in an OML) the 3OA and 4OA laws. However, we were unable to find a proof.

The transitive laws that are a consequence of (4.27) and (4.28),

$$a \overset{d}{\equiv} b = 1 \ \&\ b \overset{d}{\equiv} c = 1 \Rightarrow a \overset{d}{\equiv} c = 1 \tag{4.29}$$

$$a \overset{d,e}{\equiv} b = 1 \ \&\ b \overset{d,e}{\equiv} c = 1 \Rightarrow a \overset{d,e}{\equiv} c = 1 \tag{4.30}$$

are weaker than the 3OA and 4OA laws since both hold in lattice $\hat{\mathrm{L}}$ of Fig. 8a (which violates both laws). However, they have a weak orthoarguesian property: both fail in lattice L38m[4] (Fig. 7a) and thus cannot be derived in an OML.

The 3OA law and its equivalents have been so far (to the authors' knowledge) the only published three-variable equations derived from the 4OA law that do not hold in all OMLs. Below we show another three-variable consequence of the 4OA law that is independent of the 3OA law.

Theorem 4.11. In any OML, the three-variable condition

$$(a \to_1 d) \cap (a \overset{c,d}{\equiv} a') \le a' \to_1 d \tag{4.31}$$

holds in a 4OA, but cannot be derived from the 3OA law nor vice versa.

[4] OML L38m is neither a 3OA nor a 3GO, and in addition violates all equations we have tested that are known not to hold in all OMLs. It has been useful as a counterexample for disproving equations conjectured to hold in all OMLs. OML L42 is a 4OA, a 5OA (Section 5), and an nGO (for $n \le 9$, the upper limit we have tested) but violates all equations we have tested that are known to hold in neither 5OA nor 9GO. It has been useful for disproving equations conjectured to hold in at least one of these varieties.

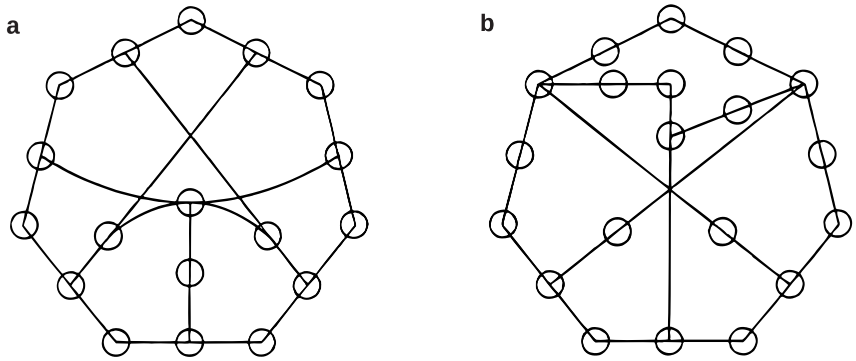

Fig. 7. Greechie diagrams for (a) OML L38m and (b) OML L42.

Proof. This condition is obviously a substitution instance of the 4OA law (4.8). On the one hand, it fails in lattice L36, but holds in lattice $\hat{\mathrm{L}}$ (Fig. 8a). On the other hand, the 3OA law (4.7) holds in lattice L36, but fails in lattice $\hat{\mathrm{L}}$. ■

An interesting OML is L38 (Fig. 8b), which violates the 4OA law, but does not violate any three-variable consequence of the 4OA law known to the authors. One possibility that came to mind was that perhaps L38 "characterizes" 4OA in an essential way, in the sense that a failure in L38 of an equation derived in a 4OA implies its equivalence to the 4OA law (analogous to the fact that a failure in O6 of an equation derived in an OML implies its equivalence to the orthomodular law). This turns out not to be the case—there is a four-variable consequence of 4OA that is strictly weaker than 4OA, but fails in L38. Whether there exists a three-variable consequence of 4OA that fails in L38 remains an open problem.

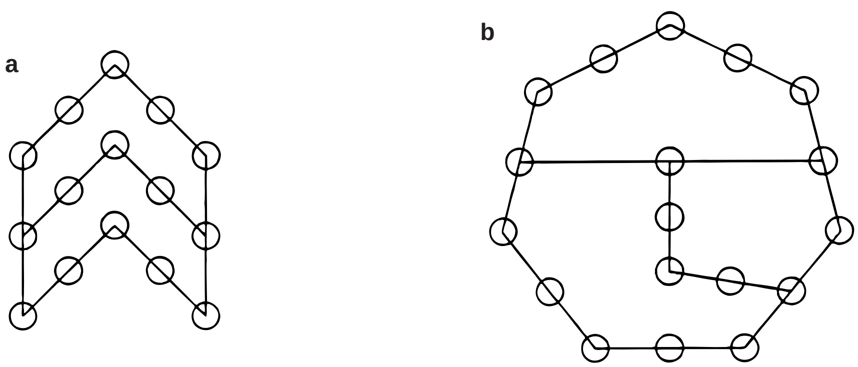

Fig. 8. Greechie diagrams for (a) $\hat{\mathrm{L}}$ from ref. 27, Fig. II, and (b) L38.

Theorem 4.12. A failure of a 4OA consequence in lattice L38 does not imply its equivalence to the 4OA law.

Proof. Writing the 4OA law as $(a' \to_1 d) \cap (a' \overset{c,d}{\equiv} b') \le b' \to_1 d$, we weaken the left-hand side of the inequality with $a \le (a' \to_1 d)$, etc. to obtain

$$\begin{aligned} &a \cap ((a \cap b) \cup (((a \cap c) \cup (a' \cap (c \to_1 d))) \\ &\quad \cap ((b \cap c) \cup ((b \to_1 d) \cap c')))) \le b' \to_1 d \end{aligned} \tag{4.32}$$

This condition fails in OML L38, but holds in L28. ∎

L38 has a peculiar history. We found it "by hand" before we had our present program for generating Greechie diagrams [17]. Later we found out that Beuttenmüller's program [8, pp. 319–328] for generating Greechie diagrams does not give L38 (and a number of other lattices). Looking for a correct algorithm, we came across McKay's isomorph-free generation of graphs. Applied to Greechie diagrams, it gave not only a correct algorithm for their generation, but also enabled writing a program which is several orders of magnitude faster than Beuttenmüller's program transcribed into the C language (originally it was written in Algol).

5. GENERALIZED ORTHOARGUESIAN EQUATIONS THAT HOLD IN $\mathcal{C}(\mathcal{H})$

Using the 3OA law as a starting point, we can construct an infinite sequence of equations $E_1, E_2, \ldots$ that are valid in all Hilbert lattices $\mathcal{C}(\mathcal{H})$. The second member E_2 of this sequence is the 4OA law and the remaining members are equations with more variables that imply the 4OA law.

Definition 5.1. We define an operation $\overset{(n)}{\equiv}$ on n variables $a_1, \ldots, a_n$ ($n \ge 3$) follows[5]:

$$\begin{aligned} a_1 \overset{(3)}{\equiv} a_2 \overset{\text{def}}{=} a_1 \overset{a_3}{\equiv} a_2 = &((a_1 \to_1 a_3) \cap (a_2 \to_1 a_3)) \\ &\cup ((a_1' \to_1 a_3) \cap (a_2' \to_1 a_3)) \end{aligned} \tag{5.1}$$

$$a_1 \overset{(4)}{\equiv} a_2 \overset{\text{def}}{=} a_1 \overset{a_4,a_3}{\equiv} a_2 = (a_1 \overset{(3)}{\equiv} a_2) \cup ((a_1 \overset{(3)}{\equiv} a_4) \cap (a_2 \overset{(3)}{\equiv} a_4)) \tag{5.2}$$

[5] To obtain $\overset{(n)}{\equiv}$ we substitute in each $\overset{(n-1)}{\equiv}$ subexpression only the two explicit variables, leaving other variables the same. For example, $(a_2 \overset{(4)}{\equiv} a_5)$ in (5.3) means $(a_2 \overset{(3)}{\equiv} a_5) \cup ((a_2 \overset{(3)}{\equiv} a_4) \cap (a_5 \overset{(3)}{\equiv} a_4))$, which means $(((a_2 \to_1 a_3) \cap (a_5 \to_1 a_3)) \cup ((a_2' \to_1 a_3) \cap (a_5' \to_1 a_3))) \cup ((((a_2 \to_1 a_3) \cap (a_4 \to_1 a_3)) \cup ((a_2' \to_1 a_3) \cap (a_4' \to_1 a_3))) \cap (((a_5 \to_1 a_3) \cap (a_4 \to_1 a_3)) \cup ((a_5' \to_1 a_3) \cap (a_4' \to_1 a_3))))$.

$$a_1 \stackrel{(5)}{\equiv} a_2 \stackrel{\text{def}}{=} (a_1 \stackrel{(4)}{\equiv} a_2 \cup ((a_1 \stackrel{(4)}{\equiv} a_5) \cap (a_2 \stackrel{(4)}{\equiv} a_5)) \tag{5.3}$$

$$a_1 \stackrel{(n)}{\equiv} a_2 \stackrel{\text{def}}{=} (a_1 \stackrel{(n-1)}{\equiv} a_2) \cup ((a_1 \stackrel{(n-1)}{\equiv} a_{\text{n}}) \cap (a_2 \stackrel{(n-1)}{\equiv} a_n)) \qquad n \geq 4. \tag{5.4}$$

Then we have the nOA *laws*

$$(a_1 \to_1 a_3) \cap ((a_1 \stackrel{(n)}{\equiv} a_2) \leq a_2 \to_1 a_3 \tag{5.5}$$

Each nOA law can be shown to be equivalent, in an OML, to equation E_{n-2} of Theorem 5.2 below by a proof analogous to that for Theorem 4.7. Thus they all hold in $\mathcal{C}(\mathcal{H})$ and for $n \geq 4$ imply (in an OML) the 4OA law. Also, as we shall show, 5OA is strictly smaller than 4OA, providing us with a new equational variety valid in $\mathcal{C}(\mathcal{H})$ that apparently has not been previously known. It remains an open problem whether in general nOA is strictly smaller than $(n - 1)$OA.

For the following theorem we will refer to the 3OA and 4OA laws in their four- and six-variable forms (4.26) and (4.17). Starting with the 3OA law, we construct a sequence of conditions as follows.

Theorem 5.2. Let $E_1, E_2, \ldots$ be the sequence of conditions constructed as follows. The first E_1 is the 3OA law expressed as

$$\begin{aligned} & a_0 \perp b_0 \;\&\; a_1 \perp b_1 \\ & \quad \Rightarrow (a_0 \cup b_0) \cap (a_1 \cup b_1) \leq b_0 \cup (a_0 \cap (a_1 \\ & \qquad \cup ((a_0 \cup a_1) \cap (b_0 \cup b_1)))) \end{aligned} \tag{5.6}$$

Given condition E_{n-1}

$$\begin{aligned} & a_0 \perp b_0 \;\&\; a_1 \perp b_1 \ldots a_{n-1} \perp b_{n-1} \\ & \quad \Rightarrow (a_0 \cup b_0) \cap (a_1 \cup b_1) \cdots \cap (a_{n-1} \cup b_{n-1}) \\ & \qquad \leq b_0 \cup (a_0 \cap (a_1 \cup (\cdots (a_i \cup a_j) \cap (b_i \cup b_j) \cdots))) \end{aligned} \tag{5.7}$$

we add new variables a_n and b_n to the hypotheses and left-hand side of the conclusion. For each subexpression appearing in the right-hand side of the conclusion that is of the form $(a_i \cup a_j) \cap (b_i \cup b_j)$, $i, j < n$, we replace it with $(a_i \cup a_j) \cap (b_i \cup b_j) \cap (((a_i \cup a_n) \cap (b_i \cup b_n)) \cup ((a_j \cup a_n) \cap (b_j \cup b_n)))$ to result in condition E_n:

$$\begin{aligned} & a_0 \perp b_0 \;\&\; a_1 \perp b_1 \ldots \;\&\; a_n \perp b_n \\ & \quad \Rightarrow (a_0 \cup b_0) \cap (a_1 \cap b_1) \cdots \cap (a_n \cup b_n) \\ & \qquad \leq b_0 \cup (a_0 \cap (a_1 \cup (\cdots (a_i \cup a_j) \cap (b_i \cup b_j)) \end{aligned}$$

$$\cap\ (((a_i \cup a_n) \cap (b_i \cup b_n)) \cup ((a_j \cup a_n) \cap (b_j \cup b_n))) \cdots))) \qquad (5.8)$$

Then E_2 is the 4OA law, and E_n ($n \geq 3$) is a condition that implies the 4OA law, holds in $\mathcal{C}(\mathcal{H})$, and cannot be inferred from 4OA.

Proof. It is obvious by definition that E_2 is the 4OA law (4.17). It is also obvious that E_n ($n \geq 3$) implies E_{n-1} and thus the 4OA law: each subexpression of E_{n-1} is greater than or equal to the subexpression of E_n that replaces it.

To show that E_n holds in $\mathcal{C}(\mathcal{H})$, we closely follow the proof of the orthoarguesian condition in ref. 28. We recall that in lattice $\mathcal{C}(\mathcal{H})$, the meet corresponds to set intersection and $\leq$ to $\subseteq$. We replace the join with subspace sum $+$ throughout: the orthogonality hypotheses permit us to do this on the left-hand side of the conclusion [8, Lemma 3, p. 67] and on the right-hand side we use $a + b \subseteq a \cup b$.

Suppose x is a vector belonging to the left-hand side of (5.7). Then there exist vectors $x_0 \in a_0, y_0 \in b_0, \ldots, x_{n-1} \in a_{n-1}, y_{n-1} \in b_{n-1}$ such that $x = x_0 + y_0 = \cdots = x_{n-1} + y_{n-1}$. Hence $x_k - x_l = y_l - y_k$ for $0 \leq k, l \leq n - 1$. In (5.7) we assume, for our induction hypothesis, that the components of vector $x = x_0 + y_0$ can be distributed over the leftmost terms on theright-hand side of the conclusion as follows:

$$\cdots \subseteq \underbrace{\underbrace{b_0}_{y_0} + (\underbrace{a_0}_{x_0} \cap (\underbrace{\underbrace{a_1}_{x_1} + (\underbrace{(\underbrace{a_0 + a_1}_{x_0 - x_1}) \cap (\underbrace{b_0 + b_1}_{-y_0 + y_1 = x_0 - x_1}) \cap \underbrace{\cdots}_{x_0 - x_1}}_{x_0 - x_1} \cap \underbrace{\cdots}_{x_0 - x_1}}_{x_1 + (x_0 - x_1) = x_0}}_{y_0 + x_0 = x}$$

In particular if we eliminate the right-hand ellipses we obtain a $\mathcal{C}(\mathcal{H})$ proof of the starting condition E_1, which is the 3OA law; this is the basis for our induction.

Let us first extend (5.7) by adding variables a_n and b_n to the hypothesis and left-hand side of the conclusion. The extended (5.7) so obtained obviously continues to hold in $\mathcal{C}(\mathcal{H})$. Suppose x is a vector belonging to the left-hand side of this extended (5.7). Then there exist vectors $x_0 \in a_0, y_0 \in b_0, \ldots, x_n \in a_n, y_n \in b_n$ such that $x = x_0 + y_0 = \cdots = x_n + y_n$. Hence $x_k - x_l = y_l - y_k$ for $0 \leq k, l \leq n$. On the right-hand side of the extended (5.7), for any arbitrary subexpression of the form $(a_i \cup a_j) \cap (b_i \cup b_j)$, where $i, j < n$, the vector components will be distributed (possibly with signs reversed) as $x_i - x_j \in a_i + a_j$ and $x_i - x_j = -y_i + y_j \in b_i + b_j$. If we replace $(a_i \cup a_j) \cap (b_i \cup b_j)$ with $(a_i \cup a_j) \cap (b_i \cup b_j) \cap (((a_i \cup a_n) \cap (b_i \cup b_n)) \cup ((a_j \cup a_n) \cap (b_j \cup b_n)))$, components x_i and x_j can be distributed as

$$\underbrace{(a_i + a_j)}_{x_i - x_j} \cap \underbrace{(b_i + b_j)}_{=\ -y_i + y_j} \cap \underbrace{((\underbrace{(a_i + a_n)}_{x_i - x_n} \cap \underbrace{(b_i + b_n)}_{=\ -y_i + y_n}) + (\underbrace{(a_j + a_n)}_{-x_j + x_n} \cap \underbrace{(b_j + b_n)}_{=\ y_j - y_n}))}_{(x_i - x_n) + (-x_j + x_n) = x_i - x_j}$$

so that $x_i - x_j$ remains an element of the replacement subexpression. We continue to replace all subexpressions of the form $(a_i \cup a_j) \cap (b_i \cup b_j)$, where $i, j < n$, as above until they are exhausted, obtaining (5.8).

That E_n ($n \geq 3$) cannot be inferred from the 4OA law follows from the fact that the 4OA law holds in L46-7 (Fig. 9), whereas $(a_1 \to_1 a_3) \cap (a_1 \overset{(5)}{\equiv} a_2) \leq a_2 \to_1 a_3$ fails in it. L46-9, Fig. 9, is the only other lattice with this property among all Greechie 3-atoms-in-a-block lattices with 22 atoms and 13 blocks. L46-7 and L46-9 are most probably the smallest Greechie 3-atoms-in-a-block lattices with that property: we scanned some 80% of smaller lattices and did not find any other. ∎

Theorem 5.3. In any nOA we have

$$a_1 \overset{(n)}{\equiv} a_2 = 1 \Leftrightarrow a_1 \to_1 a_3 = a_2 \to_1 a_3 \tag{5.9}$$

This also means that $a_1 \overset{(n)}{\equiv} a_2$ being equal to one is a relation of equivalence.

Proof. The proof is analogous to the proof of Theorem 4.10. ∎

As with Theorem 4.10, there is an open problem whether $(a_1 \to_1 a_3) \cap (a_1 \overset{(n)}{\equiv} a_2) \leq a_2 \to_1 a_3$ follows from (5.9). The fact that (5.9) fails in L46-7 and L46-9 for $n = 5$ indicates that it might.

6. DISTRIBUTIVE PROPERTIES THAT HOLD IN $\mathcal{C}$ ($\mathcal{H}$)

The distributive law does not hold in either orthomodular or Hilbert lattices, as it would make them become Boolean algebras; indeed, the failure

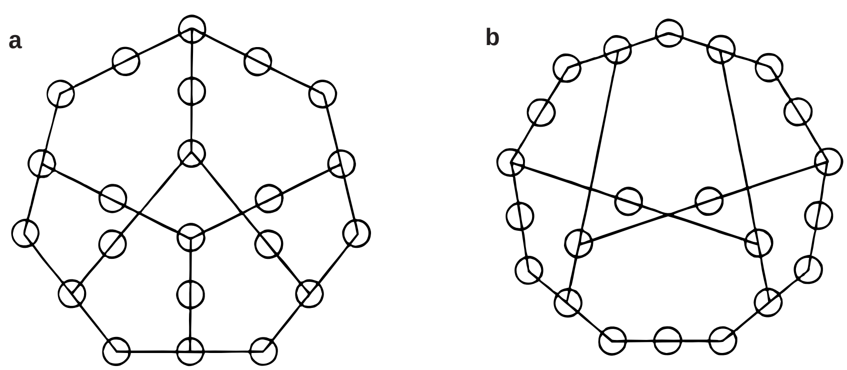

Fig. 9. Greechie diagrams for (a) L46-7 and (b) L46-9.

of this law is the essential difference between these lattices and Boolean algebras. But by using the Godowski and orthoarguesian equations that extend the orthomodular ones, we can also extend the distributive properties of OMLs such as those provided by FH. These can give us additional insight into the nature of the distributive properties of Hilbert lattices as well as provide us with additional methods for further study of these lattices. In this section we show several distributive properties that imply the Godowski and orthoarguesian equations they are derived from, meaning that they are the strongest possible in their particular form.

Definition 6.1. Let us call the following expression a *chained identity*:

$$a_1 \overset{\cdots}{\equiv} a_n \overset{\mathrm{def}}{=} (a_1 \equiv a_2) \cdots \cap (a_{n-1} \equiv a_n), \qquad n = 2, 3, 4, \ldots \tag{6.1}$$

Lemma 6.2. In any OML, the chained identity $a_1 \overset{\cdots}{\equiv} a_n$ commutes with every term (polynomial) constructed from variables $a_1, \ldots, a_n$.

Proof. In any OML, from (2.8) we have $(a \equiv b) \cap (b \equiv c) = (a \equiv b) \cap (a \equiv c)$. Using this, we rewrite $a_1 \overset{\cdots}{\equiv} a_n$ as $(a_1 \equiv a_2) \cdots \cap (a_1 \equiv a_n)$. In any OML we also have $aCa \equiv b$. Repeatedly applying the commutation law aCb & $aCc \Rightarrow aCb \cap c$, we prove $a_1 C a_1 \overset{\cdots}{\equiv} a_n$. Similarly, for any $1 \le i \le n$ we have $a_i C a_1 \overset{\cdots}{\equiv} a_n$. Repeatedly applying the commutation laws aCc & $bCc \Rightarrow a \cup bCc$ and $aCb \Rightarrow a'Cb$, we can build up $tCa_1 \overset{\cdots}{\equiv} a_n$ for any expression t constructed from variables $a_1, \ldots, a_n$.

As an exercise, the reader is invited to show an alternate proof using (3.9). ■

Theorem 6.3. In any OML in which the Godowski equation n-Go holds, the Godowski identity $a_1 \overset{\gamma}{\equiv} a_n$ commutes with any term constructed from variables $a_1, \ldots, a_n$.

Proof. Lemma 6.2 and (3.11). ■

We can use this commutation relationship in conjunction with FH to obtain immediately simple distributive laws that hold for any nGO, such as $(a_1 \overset{\gamma}{\equiv} a_n) \cap (s \cup t) = ((a_1 \overset{\gamma}{\equiv} a_n) \cap s) \cup ((a_1 \overset{\gamma}{\equiv} a_n) \cap t)$, where s and t are any terms constructed from variables $a_1, \ldots, a_n$. More general laws are also possible, as Theorem 6.6 below shows.

Lemma 6.4. In any OML the following inferences hold:

$$aCd \ \& \ bCd \ \& \ b \cap d \le c \le d \Rightarrow a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \tag{6.2}$$

$$aCd \ \& \ bCd \ \& \ b \cap d \le c \le d \Rightarrow b \cap (a \cup c) = (b \cap a) \cup (b \cap c) \tag{6.3}$$

$$aCd \ \& \ c \le d \le b' \Rightarrow c \cap (a \cup b) = (c \cap a) \cup (c \cap b) \tag{6.4}$$

Proof. For (6.2), $a \cap (b \cup c) = a \cap (b \cup (c \cap d)) =$ [using bCd, cCd] $a \cap ((b \cup c) \cap (b \cup d)) = (b \cup c) \cap (a \cap (b \cup d)) =$ [using aCd, bCd] $(b \cup c) \cap ((a \cap b) \cup (a \cap d)) =$ [using $b \cup cCa \cap b$, $a \cap bCa \cap d$] $((b \cup c) \cap (a \cap b)) \cup ((b \cup c) \cap (a \cap d)) = (a \cap b) \cup (a \cap (d \cap (b \cup c))) =$ [using dCb, dCc] $(a \cap b) \cup (a \cap ((d \cap b) \cup (d \cap c)) =$ [since $d \cap b = b \cap c$ and $c \leq d$] $(a \cap b) \cup (a \cap ((b \cap c) \cup c)) = (a \cap b) \cup (a \cap c)$.

For (6.3), $b \cap (a \cup c) \leq (b \cap (a \cup d) =$ [using bCd, aCd] $(b \cap a) \cup (b \cap d) =$ [using $b \cap d = b \cap c$] $(b \cap a) \cup (b \cap c)$. The other direction of the inequality is obvious.

For (6.4), $c \cap (a \cup b) = c \cap d \cap (a \cup b) =$ [using dCa, dCb] $c \cap ((d \cap a) \cup (d \cap b)) = c \cap ((d \cap a) \cup 0) = (c \cap a) \cup 0 = (c \cap a) \cup (c \cap b)$. ■

In passing, we note that (6.2)–(6.4) are examples of OML distributive properties that cannot be obtained directly from FH because a does not necessarily commute with either b or c (lattice MO2 would violate these conclusions). Also, the conclusion of (6.4) does not hold under the weaker hypotheses of (6.2) since the inference would fail in OML L42 (Fig. 7b). We also mention that (6.2)–(6.4) all fail in lattice O6 and thus are equivalent to the orthomodular law.

The next theorem shows examples of more general distributive laws equivalent to n-Go, where the variables a, b, and c are not necessarily equal to any other specific term and may be different from variables $a_1, \ldots, a_n$. The hypotheses of (6.6) and (6.7) can also be replaced by those of (6.8) to obtain simpler, though somewhat less general laws.

Definition 6.5. Let us call the following expression a *chained implication*:

$$a_1 \overset{\cdots}{\to} a_n \overset{\text{def}}{=} (a_1 \to_1 a_2) \cdots \cap (a_{n-1} \to_1 a_n), \qquad n = 2, 3, 4, \ldots \tag{6.5}$$

Theorem 6.6. Let t be any term constructed from variables $a_1, \ldots, a_n$. Then in any nGO ($n \geq 3$), we have the following distributive laws for any variables a, b, c not necessarily in the list $a_1, \ldots, a_n$:

$$a_1 \overset{\cdots}{\equiv} a_n \leq a \leq a_1 \overset{\cdots}{\to} a_n \;\&\; bCt \;\&\; b \cap t \leq c \leq t \;\&\; b \cup c \leq a_n \to_1 a_1$$
$$\Rightarrow a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \tag{6.6}$$

$$a_1 \overset{\cdots}{\equiv} a_n \leq a \leq a_1 \overset{\cdots}{\to} a_n \;\&\; bCt \;\&\; b \cap t \leq c \leq t \;\&\; b \cup c \leq a_n \to_1 a_1$$
$$\Rightarrow b \cap (a \cup c) = (b \cap a) \cup (b \cap c) \tag{6.7}$$

$$a_1 \overset{\cdots}{\equiv} a_n \leq a \leq a_1 \overset{\cdots}{\to} a_n \;\&\; c \leq t \leq b' \;\&\; b \cup c \leq a_n \to_1 a_1$$

$$\Rightarrow c \cap (a \cup b) = (c \cap a) \cup (c \cap b) \tag{6.8}$$

In particular, when t is $a_1 \cap a_n$ [for (6.6) and (6.7)] or a_n' [for (6.8)], an OML in which any one of these inferences holds is an nGO and vice versa.

Proof. For (6.6), let s abbreviate $a \cap (a_n \to_1 a_1)$. By (3.11) we have $(a_1 \overset{\cdots}{\to} a_n) \cap (a_n \to_1 a_1) = a_1 \overset{\gamma}{\equiv} a_n = a_1 \overset{\cdots}{\equiv} a_n$. Hence from the first hypothesis $a_1 \overset{\cdots}{\equiv} a_n = (a_1 \overset{\cdots}{\equiv} a_n) \cap (a_n \to_1 a_1) \le a \cap (a_n \to_1 a_1) \le (a_1 \overset{\cdots}{\to} a_n) \cap (a_n \to_1 a_1) = a_1 \overset{\cdots}{\equiv} a_n$, so $s = a \cap (a_n \to_1 a_1) = a_1 \overset{\cdots}{\equiv} a_n$ and by Lemma 6.2, sCt. Using (6.2), we obtain

$$a_1 \overset{\cdots}{\equiv} a_n \le a \le a_1 \overset{\cdots}{\to} a_n \ \&\ bCt \ \&\ b \cap t \le c \le t$$
$$\Rightarrow s \cap (b \cup c) = (s \cap b) \cup (s \cap c)$$

Since $b \cup c \le a_n \to_1 a_1$, it follows that $s \cap (b \cup c) = a \cap (b \cup c)$, $s \cap b = a \cap b$, and $s \cap c = a \cap c$.

In a similar way we obtain (6.7) and (6.8) from (6.3) and (6.4), respectively.

To obtain the nGO law from (6.6), we substitute $a_1 \overset{\gamma}{\equiv} a_n$ for a, $a_1 \cap a_n$ for t and c, and a_n' for b. The hypotheses of (6.6) are satisfied in any OML, and the conclusion becomes $(a_1 \overset{\gamma}{\equiv} a_n) \cap (a_n \to_1 a_1) = ((a_1 \overset{\gamma}{\equiv} a_n) \cap a_n') \cup ((a_1 \overset{\gamma}{\equiv} a_n) \cap (a_1 \cap a_n)) =$ [using (3.46)] $((a_1 \overset{\gamma}{\equiv} a_n) \cap a_1') \cup ((a_1 \overset{\gamma}{\equiv} a_n) \cap (a_1 \cap a_n)) \le a_1' \cup (a_1 \cap a_n)$, which is (3.13).

To obtain the nGO law from (6.7), we make the same substitutions as above. The conclusion becomes $a_n' \cap ((a_1 \overset{\gamma}{\equiv} a_n) \cup (a_n \cap a_1)) = (a_n' \cap (a_1 \overset{\gamma}{\equiv} a_n)) \cup (a_n' \cap a_n \cap a_1) = a_n' \cap (a_1 \overset{\gamma}{\equiv} a_n) =$ [using (3.46)] $a_1' \cap (a_1 \overset{\gamma}{\equiv} a_n) \le a_1'$. Therefore $(a_1 \cap a_n) \cup (a_n' \cap ((a_1 \overset{\gamma}{\equiv} a_n) \cup (a_n \cap a_1))) \le (a_1 \cap a_n) \cup a_1' = a_1 \to a_n$. The left-hand side evaluates as $(a_1 \cap a_n) \cup (a_n' \cap ((a_1 \overset{\gamma}{\equiv} a_n) \cup (a_n \cap a_1))) = ((a_1 \cap a_n) \cup a_n') \cap ((a_1 \overset{\gamma}{\equiv} a_n) \cup (a_n \cap a_1)) \ge ((a_1 \cap a_n) \cup a_n') \cap (a_1 \overset{\gamma}{\equiv} a_n) = a_1 \overset{\gamma}{\equiv} a_n$, establishing (3.13).

To obtain the nGO law from (6.8), we substitute $a_1 \overset{\gamma}{\equiv} a_n$ for a, a_n' for t and c, and $a_1 \cap a_n$ for b. After that the proof is the same as for (6.7). ■

In a 3OA or 4OA we can also derive distributive properties that are stronger than those that hold in OML. In fact the laws we show in Theorems 6.8 and 6.9 below are strong enough to determine a 3OA or 4OA. First we prove a technical lemma.

Lemma 6.7. In any OML we have

$$(a \to_1 c) \cap ((a \to_1 c) \cap (b \to_1 c))' \cap (a' \to_1 c) \cap (b' \to_1 c) = 0 \tag{6.9}$$

$$(a \to_1 c) \cap (((a \to_1 c) \cap (b \to_1 c))' \to_i ((a' \to_1 c) \cap (b' \to_1 c)))$$

$$\leq b \to_1 c, \qquad i = 1, 2 \tag{6.10}$$

Proof. For (6.9), by FH we have $d \cap e \cap c \cap ((d \to_1 c)' \cup (e \to_1 c)') = (d \cap e \cap c \cap (d \to_1 c)') \cup (d \cap e \cap c \cap (e \to_1 c)') = 0 \cup 0 = 0$. From (4.9) we have $d \cap e \cap (d \to_1 c) = d \cap e \cap c$. Combining these, we have $d \cap e \cap (d \to_1 c) \cap ((d \to_1 c)' \cup (e \to_1 c)') = 0$. Substituting $a' \to c$ for d and $b' \to c$ for e and simplifying with (4.12) gives the result.

For (6.10), $i = 1$: Expanding the definition of $\to_i$ ($i = 1$) and applying FH, we have $(a \to_1 c) \cap (((a \to_1 c) \cap (b \to_1 c))' \to_i ((a' \to_1 c) \cap (b' \to_1 c))) = ((a \to_1 c) \cap ((a \to_1 c) \cap (b \to_1 c))) \cup ((a \to_1 c) \cap ((a \to_1 c) \cap (b \to_1 c))' \cap (a' \to_1 c) \cap (b' \to_1 c)) =$ [using (6.9)] $((a \to_1 c) \cap (b \to_1 c)) \cup 0 \leq (b \to_1 c)$.

For (6.10), $i = 2$: Expanding the definition of $\to_2$ and applying FH, we have $(d \to_1 c) \cap (((d \to_1 c) \cap (e \to_1 c))' \to_2 (d \cap e)) = ((d \to_1 c) \cap d \cap e) \cup ((d \to_1 c) \cap (e \to_1 c) \cap (d \cap e)') =$ [using (4.9)] $(d \cap e \cap c) \cup ((d \to_1 c) \cap (e \to_1 c) \cap (d \cap e)') \leq (e' \cup (e \cap c)) \cup (e \to_1 c) = e \to_1 c$. Substituting $a' \to c$ for d and $b' \to c$ for e and simplifying with (4.12) gives the result. ■

Theorem 6.8. An OML in which

$$\begin{aligned} & d \leq a \to_1 c \;\&\; d \cap (b \to_1 c) \leq e \;\&\; e \cup f \leq a \stackrel{c}{\equiv} b \\ & \quad \Rightarrow d \cap (e \cup f) = (d \cap e) \cup (d \cap f) \end{aligned} \tag{6.11}$$

holds is a 3OA and vice versa.

Proof. Assume that (6.11) holds. Substitute $a \to_1 c$ for d, $((a \to_1 c) \cap (b \to_1 c))' \to_1 ((a' \to_1 c) \cap (b' \to_1 c))$ for e, and $((a \to_1 c) \cap (b \to_1 c))' \to_2 ((a' \to_1 c) \cap (b' \to_1 c))$ for f. It is easy to see the hypotheses of (6.11) are satisfied [use (4.14) to establish the third hypothesis]. Using (4.14), the left-hand side of the conclusion evaluates to $(a \to_1 c) \cap (a \stackrel{c}{\equiv} b)$. The right-hand side is $((a \to_1 c) \cap (((a \to_1 c) \cap (b \to_1 c))' \to_1 ((a' \to_1 c) \cap (b' \to_1 c)))) \cup ((a \to_1 c) \cap (((a \to_1 c) \cap (b \to_1 c))' \to_2 ((a' \to_1 c) \cap (b' \to_1 c))))$, which by (6.10) is $\leq (b \to_1 c) \cup (b \to_1 c) = b \to_1 c$, establishing the 3OA law (4.7).

Conversely, we show the 3OA law implies (6.11). In any OML we have from the third hypothesis $d \cap (e \cup f) \leq d \cap (a \stackrel{c}{\equiv} b)$. From the first hypothesis and the 3OA law (4.7) we obtain $d \cap a \stackrel{c}{\equiv} b \leq d \cap (b \to_1 c)$. From the second hypothesis we have $d \cap (b \to_1 c) \leq d \cap e \leq (d \cap e) \cup (d \cap f)$. Thus $d \cap (e \cup f) \leq (d \cap e) \cup (d \cap f)$. Since $d \cap (e \cup f) \geq (d \cap e) \cup (d \cap f)$ holds in any ortholattice, we conclude $d \cap (e \cup f) = (d \cap e) \cup (d \cap f)$. ■

The proof of the 4OA version of this theorem shows an application of the 3OA distributive law (6.11), where we use it to construct the inner terms of the 4OA equation.

Theorem 6.9. An OML in which

$$e \le a \to_1 d \ \& \ e \cap (b \to_1 d) \le f \ \& \ f \cup g \le a \overset{c,d}{\equiv} b$$
$$\Rightarrow e \cap (f \cup g) = (e \cap f) \cup (e \cap g) \tag{6.12}$$

holds is a 4OA and vice versa.

Proof. Assume that (6.12) holds. Since $a \overset{d}{\equiv} b \le a \overset{c,d}{\equiv} b$, we also have that (6.11) holds. So by Theorem 6.8 we have

$$(a \to_1 d) \cap (a \overset{d}{\equiv} b) \le b \to_1 d \tag{6.13}$$

By Theorem 6.8 we also have $(a \to_1 d) \cap (a \overset{d}{\equiv} c) \le c \to_1 d$, so $(a \to_1 d) \cap (a \overset{d}{\equiv} c) \cap (b \overset{d}{\equiv} c) \le (c \to_1 d) \cap (b \overset{d}{\equiv} c)$; applying Theorem 6.8 again to the right-hand side, we obtain

$$(a \to_1 d) \cap (a \overset{d}{\equiv} c) \cap (b \overset{d}{\equiv} c) \le b \to_1 d \tag{6.14}$$

In (6.12) we substitute $a \to_1 d$ for e, $a \overset{d}{\equiv} b$ for f, and $(a \overset{d}{\equiv} c) \cap (b \overset{d}{\equiv} c)$ for g. It is easy to see the hypotheses of (6.12) are satisfied, and the conclusion gives us

$$(a \to_1 d) \cap (a \overset{c,d}{\equiv} b)$$
$$= ((a \to_1 d) \cap (a \overset{d}{\equiv} b)) \cup ((a \to_1 d) \cap (a \overset{d}{\equiv} c) \cap (b \overset{d}{\equiv} c)) \tag{6.15}$$

From (6.13)–(6.15) we conclude the 4OA law (4.8).

For the converse, the proof that the 4OA law implies (6.12) is essentially identical to that for Theorem 6.8. ∎

7. CONCLUSION

Our investigation in the field of Hilbert lattices and therefore in the field of Hilbert space and its subspaces in previous sections resulted in several novel results and many decisive simplifications and unifications of the previously known results mostly due to our new algorithms for generation of Greechie lattices and automated checking of Hilbert space equations and lattice equations in general. So the results have their own merit in the theory of Hilbert space, quantum measurements, and the general lattice theory, but, as we stressed in the Introduction, we were prompted to attack the problem of generating Hilbert lattice equations and their possible connections with the quantum states (probability measures) by recent developments in the field of quantum computing. In particular, we are interested in the problem of making a quantum computer work as a quantum simulator. In order to enable this, we were looking for a way to feed a quantum computer with an algebra underlying a Hilbert space description of quantum systems. Boolean algebra underlies any classical theory or model computed on a classical computer and it imposes conditions (equations) on classical bits $\{0,1\}$ with the help of classical logic gates. For quantum theory such an algebra is still not known. Quantum computation at its present stage manipulates quantum bits $\{|0\rangle, |1\rangle\}$ by means of quantum logic gates (unitary operators) following algorithms for computing particular problems. A general quantum algebra underlying Hilbert space does exist, though. It is the Hilbert lattice we elaborated in Section 3. However, its present axiomatic definition by means of universal and existential quantifiers and infinite dimensionality does not allow us to feed a quantum computer with it. What we would need is an equational formulation of the Hilbert lattice. This would again contribute in turn to the theory of Hilbert-space subspaces, which is poorly developed. It is significant that there are two ways of reconstructing Hilbert space starting from an ortholattice. One is a pure lattice one and is presented in Section 3. The other is a pure state one [12]. The equational approach unites them.

There were only two classes of such equations known hitherto: Godowski's and Mayet's equations determined by the states defined on a lattice and four- and six-variable orthoarguesian equations determined by the projective geometry defined on it. To these we add our newly discovered (Theorem 5.2) generalized orthoarguesian equations with *n* variables. In order to interconnect and simplify the already known results on the former equations and to obtain new results we analyze the interconnections between an ortholattice and states defined on it and obtain the following results.

- By Theorems 3.10 and 3.3 the difference between classical and quantum states is that there is a single classical state for all lattice elements, while quantum states for different lattice elements are different.
- By Theorem 3.3 a classical state defined on an ortholattice turns it into the Boolean algebra.
- By Theorem 3.10 a strong state defined on an ortholattice turns it into a variety smaller than OML in which Godowski's equations hold.

On the other hand, we have:

- By Theorem 3.6 there is a way of obtaining complex, infinite-dimensional Hilbert space from the Hilbert lattice equipped with several additional conditions and without invoking the notion of state at all. States then follow by Gleason's theorem.

As for Godowski's and Mayet's equations, we obtain the following results:

- Theorems 3.12, 3.13, 3.15, and 3.16 present several new results on and simplifications of Godowski's equations based on the operation of identity given by Definition 2.4, which is also used to give a new formulation of orthomodularity by Theorem 2.8.
- New Greechie diagrams in which Godowski's equations with up to seven variables fail are presented in Figs. 3–5. They were obtained by a new algorithm for generating Greechie diagrams and a new algorithm for automated checking of passage of lattice equations through them [17] (see footnote at the end of Section 2 and the comment at the end of Section 4) and they are smaller by several atoms and blocks than the previously known ones. This makes preliminary checking of any conjecture related to Godowski's equations a lot faster.
- In Theorem 3.20 Mayet's examples, which were apparently supposed to differ from Godowski's equations, are derived from Godowski's equations.

As for the orthoarguesian equations, their consequences, and generalizations, the clue for their unification was given by three- and four-variable orthoarguesian identities (3-oa and 4-oa, defined in Definition 4.1) which enabled us to obtain the following results:

- A four-variable Eq. (4.8), the 4OA law, is equivalent to the original six-variable orthoarguesian equation as given by Day, as we showed in Theorem 4.7.
- All lower than six-variable consequences of the original orthoarguesian equation that one can find in the literature can be reduced to the three-variable Eq. (4.7), the 3OA law, as illustrated by Theorems 4.8 and 4.9.
- There is a three-variable consequence of the 4OA law which is not equivalent to the 3OA law as proved in Theorem 4.11.
- There is an n-variable generalization of the orthoarguesian equations, the nOA law, which holds in any Hilbert lattice, as proved in Theorem 5.2, and which cannot be derived from the 4OA law, as proved in Theorem 5.3.
- The nOA law added to an ortholattice turns it into a variety smaller than OML, as shown by Theorems 4.10, 4.2, and 4.3.

- Each nOA determines a relation of equivalence, as proved by Theorem 5.3.

In the end, different distributive properties that hold in the lattice of closed subspaces of any Hilbert space are given in Section 6 and several intriguing open problems are formulated following Theorems 2.9, 3.15, 3.16, 3.21, 4.3, 4.10, 4.11, and 5.3, as well as preceding Theorems 3.20 and 3.22. Open problems are also to attach a geometric interpretation to nOA and rigorously to prove that infinite-dimensional Hilbert space contains an infinite sequence of relations of equivalence. The latter claim would immediately follow from condition 5 of Theorem 3.6 if we could prove that for no n can the nOA law be inferred from the $(n-1)$OA law starting with the $n = 5$ case proved in Theorem 5.2 [28, p. 379].

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