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Unified Quantum Logic

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Unified quantum logic based on unified operations of implication is formulated as an axiomatic calculus. Soundness and completeness are demonstrated using standard algebraic techniques. An embedding of quantum logic into a new modal system is carried out and discussed.

1. INTRODUCTION

Quantum mechanics provides the semantics for quantum logic, but the question as to whether the latter can be considered a proper logic, i.e., a theory of deduction underlying quantum mechanics, is still unsettled. Two main obstables blocking the transformation of quantum logic into proper logic seemed to be the lack of a suitable operation of implication and the absence of simple semantics. Recently we proved[1] that just such a lack of a suitable implication boils down to nothing but a unique characterization of the orthomodularity. In this paper we use the result to unify all formulations of quantum logic which could be based on a particular choice of implications. We also show that a semantics for quantum logic based on a binary relation which is not determined by orthogonality[2] and related to the aforementioned characterization of orthomodularity might be conceived.

The problem of implication in quantum logic is closely related to the fact that there are many candidates for an operation of implication in the logic, none of which satisfy the proper deduction theorem. This is in con-

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trast to the situation in classical logic, where a unique operation of implication satisfies the theorem. As a result, many formulations of quantum logic have been established, all of them determined, implicitly or explicitly, by the chosen implication, even when particular formulations of quantum logic have not been conceived as logical calculuses but only as implicational algebras. Among them one can distinguish the systems employing an operation of implication simply added to already defined quantum logic, and the systems which—in order to formulate quantum logic proper—employ either the relation of implication or one of the five possible binary operations of implication expressed by means of conjunction (or disjunction) and negation and reduced to the classical operation of implication for the commensurable propositions. We shall discuss only the latter systems, but before we embark on this, we shall briefly comment on the former ones just to explain why we are going to disregard them. Such systems employ implications derived from relevance logic,[3] strict implications,[4] normal implications,[5,6] and others.[7,8] And, although these implications seem to induce neither distributivity nor modularity on quantum logic, the appropriate systems usually turn out to be nontrivial extensions of quantum logic proper, thus being too strong to serve our purpose.

As for the axiomatizations of quantum logic by means of implications, those which employ essentially the relation of implication either do not employ any operation of implication at all, or employ an improper operation which only simulates the relation of implication (e.g., such an "operation" cannot be expressed by other operations). Examples for the former axiomatization are Goldblatt's[9] binary logic (adopted from Kotas'[10] and Ackermann's[11] logic of schemata) and Nishimura's[12] sequential logic. An example for the latter axiomatization is the one put forward by Dalla Chiara[13] who based it on the so-called "trivial hook,"[8] i.e., "ultrastrict implication."[14]

Among the systems which employ proper operations of implication we can distinguish the following ones.

Mittelstaedt's[15] dialogical quantum logic, which employs the Mittelstaedt implication[16] (taken over from modular logic[17] by Mittelstaedt[18]) and is also called the quasi-implication,[19] the Sasaki implication,[8] conditional hook,[8,20,21] and conditional arrow,[22] (Clark[23] and Hardegree[24] also used this implication to formulate their axiomatic systems),

Dishkant's[25] predicate quantum calculus which employs the Dishkant implication,[20] also called the ortho-implication,[26] and

Kalmbach's[27] orthomodular propositional logic which uses the Kalmbach implication.[1]

As for the implication algebras, Finch,[28] Piziak,[29] Hardegree,[24] and Georgacarakos[20] formulated four different version of them for the Mittelstaedt implication. Georgacarakos[20] formulated the implication algebra for the relevance implication[30] (also called Kotas–Kalmbach hook[8]). And finally Abbott[31] and Georgacarakos[30] formulated two different systems for the Dishkant implication.

For such a variety of the implication characterizations of quantum logic, two main reasons were often advanced. First, it has been presupposed that one of the implications should be preferred,[14] and, secondly, most of the systems which employ operations of implication appeared to be rather complicated and impractical.[12]

We shall, however, show that, as a consequence of our recent result,[1] none of the implications is to be preferred, and that, by means of a unified operation of implication, it is possible to formulate an implication system that is as simple and practical as all those which use only the relation of implication. The system possesses a number of desirable properties, including the (weak) law of modus ponens, the (weak) law of transitivity, the property of orthomodularity derivable from the axioms of the system, a possibility of the implication to be nested, and a clear formal correspondence with an orthomodular lattice and the implications defined in it. These properties, especially the last one, suggest that such a unified operation of implication might serve to establish a property which would characterize quantum logic in a more "tractable" way than the property of orthogonality that is established on the relation of implication. As a first step toward this aim, we have employed the obtained system to embed quantum logic into a new modal logic which is weaker than Dishkant's system and which seems suitable to establish semantics, since it does not contain axioms that connect modal propositions with nonmodal ones. It turns out that the relation of accessibility corresponding to the axioms from the modal system is neither symmetric nor reflexive, and therefore such a relation of accessibility cannot determine the orthogonality relation. Thus, if this accessibility relation can be shown as not collapsing into a symmetric and reflexive one, a possibility for orthomodularity to be determined by a new relation of accessibility is opened.

2. UNIFIED QUANTUM LOGIC

The purpose of this section is to provide an axiom system[3] for quantum logic which merges all possible operations of implication of quantum

[3] The system (UQL) is an axiomatic calculus for orthomodular-valid formulas. It cannot be considered a proper logic in the usual sense of the word.

logic into a single one, and which dispenses with the relation of implication altogether. By all "possible" operations of implication we mean five operations of implication which reduce to the classical operation of implication for commensurable (compatible) propositions and which can be expressed by means of other operations, notably disjunction and negation, or conjunction and negation. Four of these were mentioned in the Introduction and the fifth, which has not served to define any system in the literature, will be designated here as the "non-tollens implication," since it is the only one which does not satisfy the law of modus tollens. We shall state them all explicitly later on.

To ease the notation while formulating the system, we shall treat the operations of implication, disjunction, and negation as primitive. Actually, by not using the relation of implication within an orthomodular lattice, as usual, but rather the operations of implications defined in it as the means of proving the lattice a model for our system, we in effect prove the following. Any of the five aforementioned operations of implication, as expressed with the help of other operations, can be substituted for any occurrence of the implication connective in our system. Such a substitution *cannot* be done with the relation of implication (the one used, e.g., in Goldblatt's system[9]).

The propositions are based on elementary propositions $p_0, p_1, p_2, \ldots$, and the following connectives: $\neg$ (negation; unary connective), $\rightarrow$ (implication; binary connective), and $\vee$ (disjunction; binary connective).

The set of propositions Q^0 is defined formally as follows:

p_j is a proposition for $j = 0, 1, 2, \ldots$

$\neg A$ is a proposition iff A is a proposition.

$A \rightarrow B$ is a proposition iff A and B are propositions.

$A \vee B$ is a proposition iff A and B are propositions.

The conjunction connective is introduced by the following definition:

$A \wedge B := \neg(\neg A \vee \neg B)$

Our metalanguage consists (apart from the common parlance) of axiom schemata from the object language as elementary metapropositions and of compound metapropositions built up by means of the following metaconnectives: & ("and"), $\veebar$ ("or"), $\sim$ ("not"), $\Rightarrow$ ("if,... then"), and $\Leftrightarrow$ ("iff"), with the usual "classical" meaning.

We define unified quantum logic UQL as the axiom system given below. The sign $\vdash$ may be interpreted as "it is asserted in UQL." Connective $\neg$ binds stronger and $\rightarrow$ binds weaker than $\vee$ and $\wedge$, and we shall occasionally omit brackets under the usual convention. To avoid a clumsy statement of the rule of substitution, we use axiom schemata instead of axioms. (From now on, whenever we mention axioms, we actually mean axiom schemata.)

Axiom Schemata.

A1. $\vdash A \to A$
A2. $\vdash A \leftrightarrow \neg\neg A$
A3. $\vdash A \to A \vee B$
A4. $\vdash B \to A \vee B$
A5. $\vdash B \to A \vee \neg A$

Rules of Inference.

R1. $\vdash A \to B \quad \& \quad \vdash B \to C \quad \Rightarrow \quad \vdash A \to C$
R2. $\vdash A \to B \quad \Rightarrow \quad \vdash \neg B \to \neg A$
R3. $\vdash A \to C \quad \& \quad \vdash B \to C \quad \Rightarrow \quad \vdash A \vee B \to C$
R4. $\vdash A \leftrightarrow B \quad \Rightarrow \quad \vdash (C \to A) \leftrightarrow (C \to B)$
R5. $\vdash A \leftrightarrow B \quad \Rightarrow \quad \vdash (A \to C) \leftrightarrow (B \to C)$
R6. $\vdash (A \vee \neg A) \leftrightarrow B \quad \Leftrightarrow \quad \vdash B$

where $\vdash A \leftrightarrow B$ means $\vdash A \to B \quad \& \quad \vdash B \to A$.

The operation of implication $A \to B$ can be any one of the following ones and no one but them[4]:

$$
\begin{array}{ll}
A \to_1 B := \neg A \vee (A \wedge B) & \text{(Mittelsteadt)} \\
A \to_2 B := \neg B \to_1 \neg A & \text{(Dishkant)} \\
A \to_3 B := (\neg A \wedge \neg B) \vee (\neg A \wedge B) & \\
\qquad\qquad \vee ((\neg A \vee B) \wedge A) & \text{(Kalmbach)} \\
A \to_4 B := \neg B \to_3 \neg A & \text{(non-tollens)} \\
A \to_5 B := (A \wedge B) \vee (\neg A \wedge B) \vee (\neg A \vee \neg B) & \text{(relevance)}
\end{array}
$$

We prove that UQL is really quantum logic by constructing the Lindenbaum–Tarski algebra for it and show that the latter is an orthomodular lattice.

By an orthomodular lattice we mean an algebra $L = \langle L^0, \perp, \cup, \cap \rangle$ such that the following conditions are fulfilled for any $a, b, c \in L^0$ and $j = 1, 2, \ldots, 5$.

L1. $a \cap b = b \cap a$
L2. $(a \cap b) \cap c = a \cap (b \cap c)$
L3. $a^{\perp\perp} = a$
L4. $a \cap a^{\perp} = 0 \qquad (a \cup a = 1)$

[4] Correspondingly, $A \vee B$ can be $(\neg A \to_{1,5} \neg B) \to_{1,5} A$, $(A \to_{2,5} B) \to_{2,5} B$, $\neg A \to_3 (\neg A \to_3 B)$, or $\neg(A \to_4 \neg B) \to_4 B$.

L5. $a \cap (a \cup b) = a$
L6. $a \cap b = (a^{\perp} \cup b^{\perp})^{\perp}$
L7(j). $a \supset_j b = 1 \quad \Leftrightarrow \quad a \leqslant b$

where $a \leqslant b := a \cap b = a \Leftrightarrow a \cup b = b$ and

$$a \supset_1 b := a^{\perp} \cup (a \cap b)$$
$$a \supset_2 b := b^{\perp} \supset_1 a^{\perp}$$
$$a \supset_3 b := (a^{\perp} \cap b^{\perp}) \cup (a^{\perp} \cap b) \cup ((a^{\perp} \cup b) \cap a)$$
$$a \supset_4 b := b^{\perp} \supset_3 a^{\perp}$$
$$a \supset_5 b := (a \cap b) \cup (a^{\perp} \cap b) \cup (a^{\perp} \cap b^{\perp})$$

The above presentation of an orthomodular lattice differs from the usual presentation, which employs

L7′. $a \cup b = ((a \cup b) \cap b^{\perp}) \cup b$

or, equivalently,[1]

L7″. $a \leqslant b \quad \& \quad a \cup b^{\perp} = 1 \quad \Rightarrow \quad b \leqslant a$

instead of L7(j). However, following Ref. 1, it is easy to show that the two presentations coincide.

That L7(j) holds in an orthomodular lattice that satisfies L1–L6 and L″ is well known.

To prove the opposite, let us assume $a \leqslant b$ and $a \cup b^{\perp} = 1$. The first assumption means $a \cap b = a$, or, equivalently, $a^{\perp} \cap b^{\perp} = b^{\perp}$. Upon introducing these expressions in the second assumption, we obtain $1 = (a \cap b) \cup b^{\perp} = b \supset_1 a$ and $1 = a \cup (a^{\perp} \cap b^{\perp}) = b \supset_2 a$ for which L7(1) and L7(2) give $b \leqslant a$. Hence, L7″ holds. Proceeding in a similar manner, we prove the same using L7(3)–L7(5).[1]

In order to prove that UQL is quantum logic, i.e., to prove that UQL has an orthomodular lattice as its model, we introduce two definitions:

Definition 1. We call $\mathscr{L} = \langle L, h \rangle$ a model of the set of formulas Q^0 if L is an orthomodular lattice and if h: UQL $\mapsto L$ is a morphism in L preserving the operations $\neg$, $\vee$, and $\rightarrow$ while turning them into $\perp$, $\cup$, and $\supset_j$ $(j = 1, \ldots, 5)$, and satisfying $h(A) = 1$ for any $A \in Q^0$ for which $\vdash A := \vdash_{\mathrm{UQL}} A$ holds.

Definition 2. We call a proposition $A \in Q^0$ true in the model $\mathscr{L}$ if for any morphism h: UQL $\mapsto L$, $h(A) = 1$ holds.

We can prove the consistency of UQL for valid formulas from L.

Theorem 1. If $\vdash A$, then A is true in any model of Q^0.

Proof. By the analogy with the binary formulation of quantum logic,[9] it is obvious that A1–A5 hold true in any $\mathscr{L}$, and that the statement is preserved by applications of R1–R3. Let us verify the remaining rules.

Ad R4. $a \supset_j b = 1 \;\;\&\;\; b \supset_j a = 1 \;\;\Rightarrow\;\; (c \supset_j a) \supset_j (c \supset_j b) = (c \supset_j b) \supset_j (c \supset_j a) = 1$. By L7(j) we transform the statement into the following one:

$$a = b \quad \Rightarrow \quad c \supset_j a = c \supset_j b$$

We have to carry out the proof for each j separately.

$\boldsymbol{j=1.}$ $a = b \;\;\Rightarrow\;\; c^\perp \cup (c \cap a) = c^\perp \cup (c \cup (c \cap b)$. It is obvious.

$\boldsymbol{j=2.}$ Obvious.

$\boldsymbol{j=3.}$ Let us assume $a = b$. Therefore $a^\perp = b^\perp$ and $c^\perp \cap a^\perp = c^\perp \cap b^\perp$. We also obtain $c^\perp \cap a = c^\perp \cap b$, as well as $c^\perp \cup a = c^\perp \cup b \Rightarrow (c^\perp \cup a) \cap c = (c^\perp \cup b) \cap c$. Combining all the three obtained expressions, we get

$$(c^\perp \cap a^\perp) \cup (c^\perp \cap a) \cup ((c^\perp \cup a) \cap c) = (c^\perp \cap b^\perp) \cup (c^\perp \cap b) \cup ((c^\perp \cup b) \cap c).$$

Hence, the statement.

$\boldsymbol{j=4}$ and $\boldsymbol{j=5.}$ These are obtained in an analogous way.

Ad R5. Starting from $a = b \Rightarrow a \supset_j c = b \supset_j c$ and copying the procedure for the previous rule, we easily prove the statement.

Ad R6. By applying L7(j) twice we obtain $a \cup a^\perp = b \Rightarrow b = 1$. Hence, R6 holds.

And finally we should prove that the definitions of implications, i.e., $\vdash (A \to B) \leftrightarrow (A \to_i B)$, $i = 1,..., 5$ turn into the corresponding definitions of $a \supset_j b$, $j = 1,..., 5$ given above. However, it is obvious and we omit the proof. ▮

To prove the opposite, i.e., the completeness of UQL for the class of valid formulas of L, we first define relation $\equiv$ and prove some related lemmas.

Definition 3. $A \equiv B := \vdash A \leftrightarrow B$.

Lemma 1. The relation $\equiv$ is a congruence relation on the algebra of propositions $\mathscr{A} = \langle Q^0, \neg, \to, \vee \rangle$, that is, for all $A, B, C, D \in Q^0$ the following hold: (1) $A \equiv A$; (2) $A \equiv B \Rightarrow B \equiv A$; (3) $A \equiv B \;\&\; B \equiv C \Rightarrow A \equiv C$; (4) $A \equiv B \Rightarrow \neg A \equiv \neg B$; (5) $A \equiv B \Rightarrow C \vee A \equiv C \vee B$; (6) $A \equiv B \Rightarrow A \vee C \equiv B \vee C$; (7) $A \equiv B \;\&\; C \equiv D \Rightarrow A \vee C \equiv B \vee D$; (8) $A \equiv B \Rightarrow C \to A \equiv C \to B$; (9) $A \equiv B \Rightarrow A \to C \equiv B \to C$; (10) $D \equiv A \;\&\; B \equiv C \Rightarrow D \to B \equiv A \to C$.

Proof. Obvious. ▮

Lemma 2. The Lindenbaum–Tarski algebra $\mathscr{A}/\equiv$ is an orthomodular lattice, i.e., L1–L7(j) are true for $\neg/\equiv$, $\vee/\equiv$, and $\rightarrow/\equiv$ turning into $^{\perp}$, $\cup$, and $\supset_j$ by means of natural morphism $k\colon \mathscr{A} \mapsto \mathscr{A}/\equiv$ which is induced by the congruence relation $\equiv$ and which satisfies $k(\neg A) = [k(A)]$, $k(A \vee B) = k(A) \vee k(B)$, and $k(A \rightarrow B) = k(A) \supset_j k(B)$. For any asserted A, i.e. for $\vdash A$, the equality $k(A) = 1$ holds.

Proof. On account of formal analogy with the binary formulation of quantum logic,[9] we consider the proofs of L1–L6 to be well known, and therefore we omit them. To prove L7(j), let us assume $\vdash A \rightarrow B$. By A1 and R3 we obtain $\vdash A \vee B \rightarrow B$ and A6 gives $\vdash B \rightarrow A \vee B$. Therefore $\vdash A \vee B \leftrightarrow B$. On the other hand, the assumption can, with the help of R6, be expressed as $\vdash (A \rightarrow B) \leftrightarrow (A \vee \neg A)$. Taken together, we obtain the following meta-implication: $\vdash (A \rightarrow B) \leftrightarrow (A \vee \neg A) \Rightarrow \vdash A \vee B \leftrightarrow B$. Thus we get $k(A) \supset_j k(B) = 1 \Rightarrow k(A) \cup k(B) = k(B)$. Hence, L7(j) holds since the other direction easily follows from L1–L6.

Let us now prove the last statement of the theorem. On account of R6, for an asserted A, i.e., for $\vdash A$, we have $k(A) = k(B) \cup [k(B)]$. Hence, the statement holds. ▮

Corollary. $\langle \mathscr{A}/\equiv, k \rangle$ is a model of theses of UQL.

Lemma 3. $k(A) = 1 \quad \Rightarrow \quad \vdash A$.

Proof. Since $k(B \vee \neg B) = 1$, we have $k(B \vee \neg B) = k(A)$, i.e., $(B \vee \neg B) \equiv A$. From it we obtain the statement by R6. ▮

Thus we have proved the completeness of UQL for valid formulas of L, that is, the following theorem.

Theorem 2. If A is true in any model of UQL, then $\vdash A$.

Two essential properties of UQL that do not appear as axioms or basic rules of inference are the (weak) law of modus ponens and the property of orthomodularity. And we shall close this section by showing their validity in UQL.

The weak law of modus ponens,

$$\vdash A \quad \& \quad \vdash A \rightarrow B \quad \Rightarrow \quad \vdash B \tag{1}$$

follows easily from R6 and R1.

That UQL possesses the property of orthomodularity is evident on account of Lemma 2 and our presentation of an orthomodular lattice. However, we consider it illustrative to give a direct proof in UQL, using the definitions of the operations of implications by means of two other operations.

We want to prove

$$\vdash A \to B \quad \& \quad \vdash \neg B \vee A \quad \Rightarrow \quad \vdash B \to A \tag{2}$$

In what follows, we shall not indicate all the steps and all the axioms and rules involved, since the proof is rather simple and straightforward.

Let us assume

$$\vdash A \to B \tag{3}$$

and

$$\vdash \neg B \vee A \tag{4}$$

(a) From (3), by R2, we get $\vdash \neg B \to \neg A$, and from this, A1, R3, and R2, we obtain $\vdash A \to \neg(\neg A \vee \neg B)$. Using A1, A4, and R1 for the first expression, and R1 for the second, we obtain

$$\vdash \neg B \to \neg B \vee (A \wedge B) \quad \& \quad \vdash A \to \neg B \vee (A \wedge B)$$

Using R3, (4), and (1), we get

$$\vdash \neg B \vee (A \wedge B) \tag{5}$$

(b) $(3) \Rightarrow \vdash A \vee B \to B \Rightarrow \vdash \neg B \to \neg(A \vee B) \Rightarrow$
$\vdash \neg B \vee A \to A \vee \neg(A \vee B) \Rightarrow (4) \,\&\, (1) \Rightarrow$
$\vdash A \vee (\neg A \wedge \neg B)$ (6)

(c) $(3) \Rightarrow \vdash \neg B \to \neg B \wedge \neg A \,\&\, \vdash \neg A \to A \wedge B \Rightarrow (4) \,\&\, (1) \Rightarrow$
$\vdash (A \wedge B) \vee (\neg B \wedge A) \vee (\neg A \wedge \neg B)$ (7)

(d) $(3) \Rightarrow \vdash A \to A \wedge B \,\&\, \vdash \neg B \to \neg A \,\&\, \vdash \neg B \to \neg B \vee A \Rightarrow$
$\vdash (A \wedge B) \vee (\neg B \wedge A) \vee ((\neg B \vee A) \wedge \neg A)$ (8)

(e) $(3) \Rightarrow \vdash \neg B \to \neg A \wedge \neg B \,\&\, \vdash A \to (\neg B \vee A) \wedge B \Rightarrow$
$\vdash (\neg A \wedge \neg B) \vee (\neg B \wedge A) \vee ((\neg B \vee A) \wedge B)$ (9)

Since the expressions (5)–(9) are the Mittelstaedt, Dishkant, relevance, non-tollens, and Kalmbach implications, respectively, by R6, R1, and once again R6, we obtain $\vdash B \to A$, i.e., the conclusion of (1). Of course, if it had not been for the illustration, it would have been enough to use but one of the definitions. (Note that, if we allow $A \to B$ to be interpreted as classical implication $\neg A \vee B$, then UQL becomes classical logic.[1])

3. A MODAL EMBEDDING

The property of orthomodularity is characterized by the relation of orthogonality, which is in turn determined by the relation of implication. The relation of orthogonality can be shown to strongly determine minimal quantum logic[9,12,13] and in a particular way quantum logic itself.[2,9,12] On the other hand, the property of orthomodularity has been shown to be nonelementary and "intractable"[2] provided it is characterized by the relation of orthogonality.

Is there any other way to characterize the property of orthomodularity and therefore quantum logic itself? To answer this question, we first have to find a relation of accessibility that characterizes quantum logic and that differs from the existing one, i.e., from the proximity relation.[9] One way to do this is to embed quantum logic into a modal logic characterized by such a relation. And in this section we carry out an embedding in a modal logic which is characterized by a relation of accessibility that includes the proximity relation[5] only as a special case. If one does not show the relation to collapse into a proximity relation within quantum logic proper, it might serve as an alternative relation of accessibility for quantum logic. This, of course, does not amount to two nonequivalent semantics for quantum logic but only to two different approaches which should eventually meet.

In doing the embedding, we shall closely follow the procedure and particular results obtained by Dishkant in Ref. 32. Dishkant carried out an embedding of quantum logic in the Brouwerian modal system (strongly determined by a proximity relation), extended so as to include the property of orthomodularity "translated"[16] in a modal axiom (or an equivalent rule of inference). Dichkant named the system Br^+.

We are going to show that it is possible to embed quantum logic in system Br^- which is weaker than Br^+. Besides, starting from L7(j), we have reduced the modal orthomodular rule of inference to a simpler equivalent rule.

We define Br^- as classical logic, i.e., the system A1–A5 & R1–R6 with $A \to B := \neg A \vee B$, to which the following axiom schemata and rules of inference are added. The sign $\models$ may be interpreted as "it is asserted in Br^-." The set of all proposition in Br^- is denoted as M^0. In Br^-, $\models \Diamond A \leftrightarrow \neg \Box \neg A$ holds.

Axiom schemata.

MA1. $\models \Box(A \to B) \to (\Box A \to \Box B)$
MA2. $\models \Box \Diamond A \to \Diamond A$
MA3. $\models \Box A \to \Box \Diamond \Box A$

[5] A proximity relation is one which is symmetric and reflexive.

Rules of inference.

MR1. $\models A \;\Rightarrow\; \models \Box A$
MR2. $\models \Box\Diamond A \to \Diamond\Box(\Diamond A \wedge \Diamond B) \;\Rightarrow\; \models \Box\Diamond A \to \Box\Diamond B$

Thus any appearance of $A \to B$ in an expression preceded by the sign $\vdash$ should be rendered as $A \to_i B$, $i = 1,..., 5$ and any appearance of it in an expression preceded by $\models$ means $\neg A \vee B$. (Our notation, which is of assertional type, differs slightly from Dihkant's[32] in the following way. Whenever Dishkant writes $\Gamma \vdash A$ or $\Gamma \models A$, i.e., "A is derivable from Γ," we write $\vdash A$ or $\models A$, i.e.,, "it is asserted," and *vice versa*. This will have no impact on our adoption of particular results from Ref. 32.)

The rule MR2 can be replaced by any of the four other possibilities corresponding to the remaining four operations of implications. We omit these, since they add nothing new to the formulation of the system.

In the following we shall use the subsequent theorems and derived rules of inference, which are either easy to prove or well known.

Theorems. MA4. $\models \Box A \to \Diamond\Box A$; MA5. $\models \Diamond\Box\Diamond A \to \Diamond A$:, MA6. $\models \Box\Diamond A \leftrightarrow \Box\Diamond\Box\Diamond A$; MA7. $\models \Box(A \wedge B) \leftrightarrow \Box A \wedge \Box B$; MA8. $\models (\Box A \vee \Box B) \to \Box(A \vee B)$; MA9. $\models \Diamond(A \wedge B) \to (\Diamond A \wedge \Diamond B)$; MA10. $\models \Diamond(A \to B) \leftrightarrow (\Box A \to \Diamond B)$.

Derived Rules of Inference. MR3. $\models A \to B \Rightarrow \models \Box A \to \Box B$; MR4. $\models \Box\Diamond A \Leftrightarrow \models \Diamond A$; MR5. $\models A \to B \Rightarrow \models \Diamond A \to \Diamond B$.

We define the embedding of UQL in Br^- by means of the following translation.

Definition 4.

$$p_k^+ := \Box\Diamond q_k \qquad (k = 1, 2,...)$$

$$(\neg A)^+ := \Box\neg A^+$$

$$(A \wedge B)^+ := \Box\Diamond(A^+ \wedge B^+)$$

$$(A \vee B)^+ := \Box\Diamond(A^+ \vee B^+),$$

where p_k and q_k are elementary propositions from UQL and Br^-, respectively, $\wedge$ in $A^+ \wedge B^+$ is the conjunction connective from Br^-, and $\wedge$ in $(A \wedge B)^+$ is from UQL, etc.

Lemma 4. Any $A^+ \in M^0$, where $A \in Q^0$, can be shown to be of the form $\Box\Diamond A^0$, where $A^0 \in M^0$.

Proof. The proof is carried out by induction on the construction of A. For p_k^+, $p_k^0 = q_k$; for $(\neg A)^+$, from the induction hypothesis it follows that $(\neg A)^+ = \Box \neg \Box \Diamond A^0$ and therefore $(\neg A)^0 = \neg \Diamond A^0$; for $(A \wedge B)^+$, $(A \wedge B)^0 = A^+ \wedge B^+$; for $(A \vee B)^+$, $(A \vee B)^0 = A^+ \vee B^+$. ∎

By Q^{0+} (from Br^-) we denote all the propositions that are translated in Br^- from UQL, and by $\mathscr{D}^-$ we denote the set of all propositions that contain elementary propositions only in the form $\Box \Diamond q$, $q \in M^0$. For them $Q^{0+} \subset \mathscr{D}^-$ holds.

Theorem 3. $\vdash A \quad \Rightarrow \quad \models A^+$.

Proof. Closely following the proof from Ref. 32, we only have to prove the subsequent lemma, which corresponds to Lemma 2 from Ref. 32. This lemma uses an equivalence relation on Q^0, which is defined as follows.

Definition 5. $A \equiv B := \models A^+ \leftrightarrow B^+$.

It can be easily proved that it is really a relation of equivalence, i.e., that it is reflexive and transitive. Also, on account of MR3 (for negation) and MA7 (for conjunction), it follows that it is a relation of congruence. Thus we can consider a natural homomorphism $e\colon \mathscr{A} \mapsto \mathscr{A}/\equiv$.

Lemma 5. The algebra $\mathscr{A}/\equiv$ is an orthomodular lattice.

Proof. Let $a = e(A)$, $b = e(B)$, and $c = e(C)$. We have to check L1–L7(j).

Ad L1 &L2. Obvious.

Ad L3. $\models A^+ \leftrightarrow A^+$ can be written, according to Lemma 4, as $\models \Box \Diamond A^0 \leftrightarrow A^+$ and on account of MA6 we obtain $\models \Box \neg \Box \neg A^+ \leftrightarrow A^+$, i.e., $\models (\neg \neg A)^+ \leftrightarrow A^+$. Hence, $a^{\perp\perp} = a$.

Ad L4. From MA2 we get $\models \neg \Box \Diamond A^+ \vee \neg \Box \neg A^+$. By Lemma 4, MA5, and A1 we obtain $\models (\neg A^+ \vee \neg \Box \neg A^+) \leftrightarrow (\neg B^+ \vee \neg \Box \neg B^+)$. Hence, $a \cap a^\perp = b \cap b^\perp$, by MR5 and MR3.

Ad L5. $\models A^+ \wedge \Box \Diamond (A^+ \vee B^+) \rightarrow A^+$ is obvious, as well as $\models A^+ \rightarrow \Box \Diamond A^+ \vee \Box \Diamond B^+ \Rightarrow$ (MA8) $\Rightarrow \models A^+ \rightarrow \Box \Diamond (A^+ \vee B^+) \Rightarrow \models A^+ \rightarrow A^+ \wedge \Box \Diamond (A^+ \vee B^+)$. Thus $\models A^+ \wedge \Box \Diamond (A^+ \vee B^+) \leftrightarrow A^+ \Rightarrow$ (MR3, MR5) $\Rightarrow \models \Box \Diamond (A^+ \wedge (A \vee B)^+) \leftrightarrow \Box \Diamond A^+ \Rightarrow$ (Lemma 4, MA6) $\Rightarrow$ $\models (A \wedge (A \vee B))^+ \leftrightarrow A^+ \Rightarrow a \cap (a \cup b) = a$.

Ad L6. Starting with $\models \neg \Diamond (\Box \neg A^+ \vee \Box \neg B^+) \leftrightarrow \Box (\Diamond A^+ \wedge \Diamond B^+)$ and applying MA7, Lemma 4, MR5, and MR3, we arrive at L6.

Ad L7(j). It suffices to prove it for but one j, e.g., $j=1$.
Let us start with the premise:

$$\models \Box\Diamond(\Diamond A^+ \to \Box\Diamond(A^+ \wedge B^+)) \leftrightarrow \Box\Diamond(\Box\neg A^+ \vee A^+)$$

The direction from left to right is a tautology, since $\models \Diamond(\Box\neg A^+ \vee A^+)$ is a tautology (MA2 by MA10). Since the expression with which we started is the antecedent of the meta-implication we have to prove, we only need to consider the opposite direction. Taking into account that $\models \Box\Diamond(\Box\neg A^+ \vee A^+)$ is a tautology, by modus ponens, MR4, M10, Lemma 4, $\models \Diamond\Box\Diamond\Box A \leftrightarrow \Diamond\Box A$, MA6, MR2, once again Lemma 4, and MA6, we get $\models A^+ \to B^+$. Therefore $\models A^+ \vee B^+ \leftrightarrow B^+$. By MR5, MR3, Lemma 4, and MA6, we obtain the needed consequent: $\models \Box\Diamond(A^+ \vee B^+) \leftrightarrow B^+$. Hence, $a^\perp \cup (a \cap b) = a \supset_1 b = 1 \Rightarrow a \leqslant b$.
This completes the proof of the lemma. ∎

The rest of the proof of the theorem remains the same as in Ref. 32, and we refer the reader to it. ∎

To prove the completeness, we shall also often refer to the unaffected parts of an analogous proof in Ref. 32.
Let us first introduce a definition and prove some lemmas.

Definition 6. For any morphism h: UQL $\mapsto L$ a forcing relation $\Vdash$ between elements of $L^- = L^0 - \{0\}$ and formulas of $\mathscr{D}^-$ is defined as follows:

(a) $a \Vdash \Box\Diamond q \quad \Leftrightarrow \quad a \leqslant h(q)$
(b) $a \Vdash \neg A \quad \Leftrightarrow \quad \sim(a \Vdash A)$
(c) $a \Vdash A \wedge B \quad \Leftrightarrow \quad a \Vdash A \quad \& \quad a \Vdash B$
(d) $a \Vdash \Box A \quad \Leftrightarrow \quad \forall b(b \Vdash A \quad \veebar \quad b \leqslant a^\perp)$,

where q is an elementary proposition from Br$^-$, $A, B \in \mathscr{D}^-$, $a, b \in L^-$.

Lemma 6. Let $A \in Q^0$ and $a \in L^-$. Then $a \Vdash A^+ \Leftrightarrow a \leqslant h(A)$.

Proof. The proof is carried out by induction on the construction of A. For the case $A = \neg B$, we refer to Ref. 32. Let us consider the remaining case $A = B \wedge C$. By (d) and (c) of Def. 6, $a \Vdash (B \wedge C)^+$ transforms into

$$\forall b[\forall c((c \Vdash A^+ \quad \& \quad c \Vdash B^+) \quad \Rightarrow \quad c \leqslant b^\perp) \quad \Rightarrow \quad b \leqslant a^\perp] \tag{10}$$

In this expression we can substitute $c \Vdash A^+$ and $c \Vdash B^+$ by $c \leqslant h(A)$ and $c \leqslant h(B)$, respectively, on account of the inductive assumption. Then we can use the statement $\forall c(c \leqslant a \Rightarrow c \leqslant b) \Leftrightarrow a \leqslant b$, which holds for our lattice, to reduce expression (10) on $a \leqslant h(A) \cap h(B)$, which is the consequent we need. Proceeding in the opposite direction, we prove the lemma. ∎

Lemma 7. Let us say $A \in \mathscr{D}^{0-}$ iff A is a conjunction of several propositions belonging to $\mathscr{D}^{-} \cap \mathscr{L}$, when $\mathscr{L}$ is the set of apodictic propositions. (Of course, $\mathscr{D}^{-} \cap \mathscr{L} \subset \mathscr{D}^{0-} \subset \mathscr{D}^{-}$.) Then, for any $A \in \mathscr{D}^{0-}$, there is $[A] \in L^{0}$ such that for any $a \in L^{-}$, $a \Vdash A \Leftrightarrow a \leqslant [A]$.

Proof. The proof is established by induction on the construction of A. In our case, (a) from Definition 6 is the basis of the induction. The rest of the proof is the same as in Ref. 32, and we refer the reader to it. ∎

Lemma 8. Let $C \in \mathscr{D}^{-}$ and C be one of the axioms of Br^{-} or $C = \Box\Diamond\Box\Diamond q \to \Box\Diamond q$, where q is an elementary proposition from Br^{-}. Then $a \Vdash C$ for any $a \in L^{-}$.

Proof. For all the axioms (more precisely: axiom schemata) we refer to Ref. 32. For, if axioms from Br^{+} are forced, then axioms from Br^{-} must be forced too, since the latter are special cases of the former. Thus, we only have to prove $a \Vdash \Box\Diamond\Box\Diamond q \to \Box\Diamond q$ instead of (v) of Lemma 6 of Ref. 32. The proof remains unchanged, except that we have to take $[\Box\Diamond q]$ instead of $[\Box q]$. ∎

Lemma 9. Any rule of inference of Br^{-} has the following property: If its premises are forced by all $a \in L^{-}$, then its conclusion is forced by all a.

Proof. For all rules of inference apart from MR2 we refer the reader to Ref. 32. The rule MR2 is a special case of the following rule:

$$\models \Box A \to \Diamond\Box(A \wedge B) \quad \Rightarrow \quad \models \Box A \to \Box B \tag{11}$$

To prove the lemma, it is enough to prove its equivalence (in Br^{+}) with the following modal form of the orthomodularity property (corresponding to L7″),

$$\models \Box A \to \Box B \quad \& \quad \models \Box B \to \Diamond\Box A \quad \Rightarrow \quad \models \Box B \to \Box A \tag{12}$$

(11) ⇒ (12). Let us assume the premises of (12). The first one is equivalent to $\models \Box A \leftrightarrow \Box A \wedge \Box B$. Upon introducing the latter in the second premise of (12), by using (11), we obtain the conclusion of (12).

(12) ⇒ (11). Let us assume the premise of (11). Since

$\models \Box A \wedge \Box B \rightarrow \Box A$ is a tautology, by (12) we obtain $\models \Box A \rightarrow \Box B \wedge \Box A$, from which the conclusion of (11) follows immediately. ▮

Lemma 10. Let $\Gamma \subset \mathscr{D}^-$, $A \in \mathscr{D}^-$, and let Δ^- be the set of all propositions of the form $\Box \Diamond \Box \Diamond q \rightarrow \Box \Diamond q$, where q is an elementary proposition and q occurs in A or in a proposition from Γ. Then, if $\models A$, there is a proof of A from $\Gamma \cup \Delta$ such that all the propositions of the proof belong to $\mathscr{D}^-$.

Proof. Let ψ be a proof of A from Γ, and let us replace all q's occurring in all propositions of ψ by $\Box \Diamond q$. Then we obtain a proof ψ' and A' from Γ'. Obviously $\psi' \subset \mathscr{D}^-$. Propositions of Γ' and A' contain q only in the form $\Box \Diamond \Box \Diamond q$. Since proposition $\Box \Diamond q \leftrightarrow \Box \Diamond \Box \Diamond q$ is a theorem (MA6) of Br^- and belongs to $\mathscr{D}^-$, we can obtain a proof Σ of A from $\Delta^- \cup \{A'\}$. We can choose Σ so that any of its propositions belong to $\mathscr{D}^-$. Analogously, for all $Z' \in \Gamma$ which belong to ψ', one can obtain their proof $\pi(Z')$ from $\Delta^- \cup Z'$. The needed proof is the union of all $\pi(Z')$, ψ', and Σ. ▮

Theorem 4. $\models A^+ \quad \Rightarrow \quad \vdash A$.

Proof. Let us consider the morphism h: $\mathrm{UQL} \mapsto L$ such that, for any B belonging to the set T of all theorems of UQL, $h(A) = 1$ holds. By lemma 6, $a \Vdash B^+$ for any $a \in L^-$ and any $B \in T$. By Lemma 10 we conclude, since $T^+ \subset \mathscr{D}^-$, that there is a proof of A^+ such that all its propositions belong to $\mathscr{D}^-$, and from Lemmas 8 and 9 it follows that any proposition belonging to the proof is forced by any $a \in L^-$. Thus $a \Vdash A^+$ and, by Lemma 6, $h(A) = 1$. Hence, $\vdash A$. ▮

4. DISCUSSION

In Sec. 2 we have formulated quantum logic as an implication calculus UQL. A unified operation of implicaıtion that satisfies the system includes all proper operations of implication (see Sec. 1) and serves to formulate quantum logic in such a way that the relation of implication cannot. On the other hand, quantum logic has been shown to be "intractable,"[2] provided it is characterized by the relation of orthogonality, which is in turn determined by the relation of implication. Does this mean that we could expect a better "tractability" of quantum logic if it were characterized by an operation of implication instead of the relation? An answer to this question is not known. If it is possible to characterize quantum logic by an accessibility relation which includes a symmetric and relexive one only as a special case, then an operation of implication seems to be the most appropriate candidate for establishing such a characterization. One way to find such a relation of accessibility is to embed quantum logic in a modal logic characterized by the relation. And this is what we have done in Sec. 3. We carried out an embedding of quantum logic into an extension of modal system K,[34] obtained by adding MA2, MA3, and MR2 to K, which we called Br^-. System $\mathrm{Br}^0 = \mathrm{K} + \mathrm{MA2} + \mathrm{MA3}$ is characterized by the accessibility relation R which satisfies the following conditions[35,36]:

(i) $\forall w_1\, \exists w_2 [w_1 R w_2 \;\&\; \forall w_3 (w_2 R w_3 \Rightarrow w_1 R w_3)]$

(ii) $\forall w_1\, \forall w_2 [w_1 R w_2 \Rightarrow \exists w_3 (w_2 R w_3 \;\&\; \forall w_4 (w_3 R w_4 \Rightarrow w_1 R w_4))]$

Condition (ii) is exactly the one which Dishkant used to establish semantics for minimal quantum logic.[37] However, he also used reflexivity instead of condition (i) and a reflexive accessibility relation corresponds to axiom T ($\models \Box A \rightarrow A$). Besides, while establishing the equivalence of algebraic and semantic models, Dishkant reduced his relation of accessibility to the proximity relation (using the orthogonality relation to define R). Actually, it is only to be expected since minimal quantum logic is strongly determined by a class of orthoframes defined by the orthogonality relation. Does this mean that the appearance of a different logic $\mathrm{Br}^-(\mathrm{Br}^0)$ in which we can embed quantum logic (minimal quantum logic) is related only to a different translation we used in Def. 4 in contradistinction to $(A \wedge B)^+ := A^+ \wedge B^+$ used in Refs. 9 and 32?[6] The answer is in the negative since we are also able to embed quantum logic into a modal system $\mathrm{Br}^- + \models \Box \Diamond \Box A \rightarrow \Box A$ using not Def. 4 but the definition from Ref. 9.[38] In this case the condition (ii) for the appropriate relation of accessibility has to be strengthened but R is not reduced to the proximity relation. Of course, for minimal quantum logic the propositions translated in KTB and those translated in $\mathrm{Br}^0 + \models \Box \Diamond \Box A \rightarrow \Box A$ coincide and the relation of accessibility *for the propositions* boils down to the proximity relation. Whether an analogous reduction happens for quantum logic proper is not known. Namely, as soon as we add the property of orthomodularity to minimal quantum logic we no longer know whether the resulting quantum logic proper is still strongly determined by a class of orthomodular orthoframes or not.

In any case, it follows that we cannot infer the properties of (minimal) quantum logic directly from the properties of a modal logic in which (minimal) quantum logic can be embedded. What we, therefore, can do is to use

[6] The two definitions are equivalent in KTB but not in Br^0. We can compare the embedding of minimal quantum logic in KTB vs. Br^0 with the embedding of classical logic in S5 when translation $A^+ := \Box A^+$ is applied vs. its embedding in S4 when $A^+ := \Box \Diamond A^+$ is applied.[39] ($K := \mathrm{MA1}$; $B := \models A \rightarrow \Box \Diamond A$).

such modal systems as indicators for possible semantics of quantum logic. This, of course, does not amount to saying that more non-equivalent semantics for quantum logic could exist but only that there are presumably more approaches to the problem which should eventually meet.

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