MATRIX MODEL ↔ INVERSE-SQUARE INT. MODEL

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QHE $\propto$ NC U(1) CS theory with $M_{\infty\times\infty}$ $(\#e^- = \infty)$

$$\begin{array}{rcl} \textbf{QHE} & \longleftrightarrow & \textbf{CS MM}_{\mathrm{N\times N}} \\ \textbf{CS MM}_{\mathrm{N\times N}} & \longleftrightarrow & \textbf{Inverse} - \textbf{square interaction model} \\ \textbf{AdS} & \longleftrightarrow & \mathbf{2+1} \textbf{ CS fermions} \end{array}$$

MM (BIPZ)

$$\mathcal{L} = Tr\dot{M}^2 + TrV(M)$$

$$\begin{aligned} M &= U\Lambda U^\dagger \\ \dot{M} &= U\left(\dot{\Lambda} + [U^{-1}\dot{U}, \Lambda]\right)U^\dagger \end{aligned}$$

Conserved quantity:

$$J = [M, \dot{M}] = U\left([\Lambda, \dot{\Lambda}] + [\Lambda, [U^{-1}\dot{U}, \Lambda]]\right)U^\dagger$$

$$Tr\dot{M}^2 + TrV(M) = Tr\dot{\Lambda}^2 + \sum_{a\neq b} \frac{(U^{-1}JU)_{ab}(U^\dagger JU)_{ba}}{(\lambda_a - \lambda_b)^2} + TrV(M)$$

$$\mathbf{0+1}\ \textbf{MQM} \;\rightarrow\; \mathbf{1+1}\ \textbf{NQM} \qquad \textbf{fermionization}$$

$$M = \begin{pmatrix} \lambda_1 & \dots & \\ & & \\ & & \lambda_N \end{pmatrix} \;\rightarrow\; x_1 \qquad\qquad x_N$$

Only for M with invariance:

M=R real symmetric O(N) invariant
M=H complex hermitean U(N) invariant
M=Q quaternionic real Sp(N) invariant

The collective-field formulation of the MM

$$\mathcal{L} = Tr\dot{M}^2 + TrV(M) \qquad \begin{array}{ll} H & \text{A. Jevicki, B. Sakita} \\ Q & \text{I.A., H. Levine, A. Jevicki} \end{array}$$

Collective field $\longrightarrow$ Bosonozation in N$\longrightarrow \infty$

$$\phi(k,t) = Tre^{-ikM(t)}, \qquad \phi(x,t) = \int \frac{dk}{2\pi} e^{-ikx}\phi(k,t)$$

Hamiltonian

$$-T_+ = \frac{1}{2}\int dxdy\pi(x)\Omega_{xy}(\phi)\pi(y) - \frac{i}{2}\int dx\omega(\phi)\pi(x) \ \ \& \ \ J(\phi)$$

Quaternionic (self-dual) matrix

$$Q = \begin{pmatrix} Q_0 + iQ_1 & -Q_2 - iQ_3 \\ Q_2 - iQ_3 & Q_0 - iQ_1 \end{pmatrix} = Q_\alpha e_\alpha; \alpha = 0, \ldots, 3$$

- $Q^\dagger = Q$, real eigenvalues
- group properties
- diagonalized by Sp(N)

$$Q_0 = Q_0^{ij} h_{ij}^+, \ Q_l = Q_l^{ij} h_{ij}^-, \ l = 1,2,3, \ [h_{ij}^\pm]_{mn} = \delta_{im}\delta_{jn} \pm \delta_{in}\delta_{jm}$$

$$\begin{aligned} \Omega_{xy}(\phi) &= \frac{1}{2}\left[\sum_{i\leq j}^N (1+\delta_{ij})\frac{\partial\phi(x)}{\partial Q_0^{ij}}\frac{\partial\phi(y)}{\partial Q_0^{ij}} + \sum_{l=1}^3\sum_{i<j}^N \frac{\partial\phi(x)}{\partial Q_l^{ij}}\frac{\partial\phi(y)}{\partial Q_l^{ij}}\right] \\ &= \partial_x\partial_y\delta(x-y)\phi(x) \end{aligned}$$

$$\begin{aligned} \omega_x(\phi) &= -\frac{1}{2}\left(\sum_{i\leq j}^N (1+\delta_{ij})\frac{\partial^2\phi(x)}{\partial Q_0^{ij\,2}} + \sum_{l=1}^3\sum_{i<j}^N \frac{\partial^2\phi(x)}{\partial Q_l^{ij\,2}}\right) \\ &= (\lambda-1)\partial_x^2\phi(x) + 2\lambda\partial_x\phi\int\frac{\phi(y)dy}{x-y} = \partial_x\left(\phi(x)\partial_x\frac{\partial \ln J}{\partial\phi(x)}\right) \end{aligned}$$

Conformal invariance of the collective-field theory

$$H = \frac{1}{2}\int dx\phi(x)(\partial_x\pi)^2 + \int dx\phi(x)V(x) + \\ + \frac{1}{2}\int dx\phi(x)\left(\frac{\lambda-1}{2}\frac{\partial_x\phi(x)}{\phi(x)} + \lambda \fint \frac{\phi(y)dy}{x-y}\right)^2 + \dots$$

Corresponding Lagrangian (free part)

$$\mathcal{L}(\phi,\dot{\phi}) = \frac{1}{2}\frac{(\partial_x^{-1}\dot{\phi})^2}{\phi} - \frac{1}{2}\phi\left(\frac{\lambda-1}{2}\frac{\partial_x\phi}{\phi} + \lambda\fint dy\frac{\phi(y)}{x-y}\right)^2$$

Infinitesimal transformations - $t' = t - \epsilon t^n$

$$\delta\phi = \phi'(x,t) - \phi(x,t) = (-d_\phi n t^{n-1} + d_x n t^{n-1} x\partial_x + t^n\partial_t)\phi(x,t)$$

Conserved charges Q_n

$$Q = \int dx\frac{\delta\mathcal{L}}{\delta\dot{\phi}}\delta\phi - A(\phi,\dot{\phi}),\ \delta S = \int dt\frac{dA(\phi,\dot{\phi})}{dt}$$

$$Q_n = \frac{n(n-1)}{4}\int dx x^2\phi - \frac{nt^{n-1}}{2}\int dx x\phi\partial_x\pi + t^n H$$

$$\begin{aligned} Q_0 &= H \\ Q_1 &= -\frac{1}{2}\int dx x\phi\partial_x\pi + tH \\ Q_2 &= \frac{1}{2}\int dx x^2\phi - t\int dx x\phi\partial_x\pi + t^2 H \end{aligned}$$

$$\begin{aligned} Q_0(t=0) &\to T_+ \\ Q_1(t=0) &\to T_0 \qquad SU(1,1):\ [T_+,T_-] = -2T_0 \\ Q_2(t=0) &\to T_- \qquad\qquad\qquad [T_0,T_\pm] = \pm T_\pm \end{aligned}$$

BPS limit

$$H = \frac{1}{2}\int dx\rho(\partial_x\pi)^2 + \frac{1}{2}\int dx\rho\left(\frac{\lambda-1}{2}\frac{\partial_x\rho}{\rho} + \lambda\fint\frac{\rho(y)dy}{x-y}\right)^2 - \int\frac{\delta\omega}{\delta\rho}$$

Ground state (BPS)
Large N limit $\longrightarrow$ 1^{st} order Bogomol'nyi type eq.

$$\frac{\lambda-1}{2}\frac{\partial_x\rho}{\rho} + \lambda\fint\frac{\rho(y)dy}{x-y} = 0, \quad \rho = \rho_0 = \mathrm{cte}\ \forall\lambda$$

Ground state is degenerate in leading N!

$$\Rightarrow \frac{1}{2}\int dx\rho\left(\frac{\lambda-1}{2}\frac{\partial_x\rho}{\rho} + \lambda\fint\frac{\rho(y)dy}{x-y} + \frac{v}{x}\right)^2 =$$
$$= \frac{1}{2}\int dx\rho\left(\frac{\lambda-1}{2}\frac{\partial_x\rho}{\rho} + \lambda\fint\frac{\rho(y)dy}{x-y}\right)^2 + \frac{v(v-1+\lambda)}{2}\int\frac{\rho(x)}{x^2} +$$
$$+\frac{v\lambda}{2}\left(\int\frac{\rho(x)}{x}\right)^2 + \frac{v\lambda\pi^2}{2}\rho(0)$$

If $\exists\rho;\ \rho(0) = 0$ & $\rho(x) = \rho(-x)$ for $v = 1-\lambda$
Bogomol'nyi type eq.

$$\frac{\lambda-1}{2}\frac{\partial_x\rho}{\rho} + \lambda\fint\frac{\rho(y)dy}{x-y} + \frac{v}{x} = 0$$

$$\rho_s(x) = \rho_0\frac{x^2}{x^2+b^2},\ \rho_0 b = \frac{1-\lambda}{\lambda}$$

Quantum corrections $\rho(x) = \rho_0 + \eta(x)$
Spectrum known for $\rho(x) = \mathrm{cte}$, $\rho(x) = \rho_s(x)$

The BPS solution - solitons (the static tachyonic background)
The non-BPS solutions - moving solitons

The solitons of the collective field description are dual quasi-particles.

Introduce a new field $m(x,t)$ describing quasi-particles in the prefactor of the wave functional:

$$V^{\kappa}[\rho,m] = \exp\left\{\kappa \int\int dx dy \rho(x) \ln|x-y| m(y)\right\}$$

The duality is displayed by the following relations:

$$T_{+}[\rho,\lambda]V^{\kappa}[\rho,m] = \left[-\frac{\lambda}{\kappa}T_{+}[m,\kappa^2/\lambda] + \right.$$
$$+\frac{\kappa\pi^2}{2}\int dx \rho(x)m(x)(\lambda\rho(x)+\kappa m(x)) +$$
$$\left. +\frac{(\lambda+\kappa)(\kappa-1)}{2}\int\int dx dz \frac{m(z)\partial_x\rho(x)}{x-z}\right] V^{\kappa}[\rho,m]$$

$$T_0[\rho,\lambda]V^{\kappa}[\rho,m] = -\left\{T_0[m,\kappa^2/\lambda] + \right.$$
$$\left. +\frac{1}{2}E_0(N,\lambda) + \frac{1}{2}E_0(M,\kappa^2/\lambda) + \frac{1}{2}\kappa N M\right\} V^{\kappa}[\rho,m]$$

New su(1,1) generators for particles and solitons

$$\mathcal{T}_{+} = T_{+}[\rho,\lambda] + \frac{\lambda}{\kappa}T_{+}[m,\kappa/\lambda] + \mathcal{H}_{int}$$
$$\mathcal{H}_{int} = -\frac{(\lambda+\kappa)(\kappa-1)}{2}\int\int dx dz \frac{m(z)\partial_x\rho(x)}{x-z} +$$
$$-\frac{\kappa\pi^2}{2}\int dx \rho(x)m(x)(\lambda\rho(x)+\kappa m(x))$$

$$\mathcal{T}_0 = T_0[\rho,\lambda] + T_0[m,\kappa^2/\lambda]$$
$$\mathcal{T}_- = T_-[\rho,\lambda] + \frac{\kappa}{\lambda} T_-[m,\kappa^2/\lambda]$$

We interpret the operator $\mathcal{T}_+$ as a non-hermitian Hamiltonian for particles and solitons. After hermitisation of $\mathcal{T}_+$ we obtain the hermitian form

$$\mathcal{H}^M = H[\rho,\lambda] + \frac{\lambda}{\kappa} H[m,\kappa^2/\lambda] + \mathcal{H}_{int}$$

where $H[\rho,\lambda]$ is

$$\begin{aligned} H[\rho,\lambda] = {}& \frac{1}{2}\int dx \rho(x)(\partial_x \pi)^2 + \\ & + \frac{1}{2}\int dx \rho(x) \left(\frac{\lambda-1}{2}\frac{\partial_x \rho(x)}{\rho} + \lambda \fint dy \frac{\rho(y)}{x-y} \right)^2 + \\ & - \mu \int dx \rho(x) + \\ & - \frac{\lambda-1}{4}\int dx \partial_x^2 \delta(x-y)|_{y=x} - \frac{\lambda}{2}\fint dx \partial_x \frac{1}{x-y}|_{y=x} \end{aligned}$$

and $H[m,\kappa^2/\lambda]$ is obtained from $H[\rho,\lambda]$ substituing m for ρ and κ^2/λ for λ.