Renormalisability of 2PI-Hartree approximation to scalar field theories

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Plan of the talk:

- Motivation: Extended Higgs-sector of the Standard Model,
  Improved approximation schemes to effective meson theories
- The 2PI-approximation to the effective action (quick review)
  The 2PI-Hartree truncation
- General analysis of the renormalisability of the 2PI-Hartree approximation
  Examples: the $O(N)$ model with one and with two N-plets

Increasing role of scalar fields in particle physics

Inflaton, dark matter, quintessence (cosmological acceleration)

Phantom/Shadow fields: not coupled to SM force-fields and fermion-matter, but could couple to SM-Higgs:
Higgs portal to the phantom world (Patt, Wilczek (2006))

$$V(\phi_p,\Phi_s)=\mu_s^2\Phi_s^\dagger\Phi_s+\lambda(\Phi_s^\dagger\Phi_s)^2+\mu_p^2\phi_p^\dagger\phi_p+\lambda_p(\phi_p^\dagger\phi_p)^2-\eta\Phi_s^\dagger\Phi_s\phi_p^\dagger\phi_p$$

General symmetry breaking pattern: $\Phi_s\to v_s+H_s, \phi_p\to v_p+H_p$
Consequences (weak coupling analysis):
Mixing of standard and phantom fields in mass eigenstates
Invisible Higgs decays

Novel $v_p=0$ mechanism for generation of electroweak symmetry breaking
(generalised Coleman-Weinberg phenomenon)

General non-perturbative analysis:
Zs. Szép, A.P., Phys. Lett. **B642** (2006) 384
Europhys. Lett. **79** (2007) 51001

Similar studies (a very partial list):

W.F. Chang, J.N. Ng and J.M.S. Wu, Phys. Rev. **D74** (2006) 095005
ibid. **D75** (2007) 115016
D.G. Cerdeno, A. Dedes and T.E.J. Underwood, JHEP **0609** (2006) 067
X. Calmet and J.F. Oliver, Europhys. Lett. **77** (2007) 51002
J.A. Casas, J.R. Espinosa and I. Hidalgo, Nucl. Phys. **B777** (2007) 226
J.R. Espinosa and M. Quiros, hep-ph/0701145
M. Bowen, Y. Cui and J.D. Wells, JHEP **0703** (2007) 036
O. Bahat-Treidel, Y. Grossman and Y. Rozen, JHEP **0705** (2007) 022
T. Hambye and M.H.G. Tytgat, hep-ph/0707.0633
O. Bertolami and R. Rosenfeld, hep-ph/0708.1784

All based on
perturbative weak coupling analysis of the Higgs-shadow world coupling.

Interest of applying non-perturbative approaches
(Dyson-Schwinger, 2PI, large N, etc.)

Effective meson theories of low energy

Continued interest in applications to the phase diagram of strong matter
see reviews by

Zs. Szép, PoS JHW2005:017,2006
R. Casalbuoni, PoS CPOD2006:001,2006

For refined applications of the linear sigma model see P. Kovács's talk

Attempt to apply 2PI-approximation:
D. Röder, J. Rupert and D.H. Rischke, Phys. Rev. **D68** (2003) 016003
D. Röder, J. Rupert and D.H. Rischke, Nucl. Phys. **775** (2006) 127
Quotation prompting this investigation:
"Renormalisation of many-body approximation schemes is non-trivial, but does not change the results qualitatively. We therefore simply omit the vacuum contributions to the loop integrals."
Our results present evidence for:

- Transparent non-perturbative renormalisation scheme exists for 2PI-approximation truncated at the Hartree level.
- Vacuum contributions produce important quantitative modifications in the phase diagram.

Quick outline of the 2PI approximation for 1-component real scalar field

$$L(\varphi) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - U(\varphi)$$

2PI-action (Cornwall-Jackiw-Tomboulis, 1974):

$$V[\phi, G] = U(\phi) + \frac{1}{2}\int_k \ln G^{-1}(k) + \frac{1}{2}\int (D^{-1}(k,\phi)G(k) - 1) + V_2(\phi, G),$$

$$D^{-1}(k,\phi) = -k^2 + U''(\phi)$$

Equations of motion:

$$\frac{\delta V}{\delta\phi(k)} = 0, \qquad \frac{\delta V}{\delta G(k)} = 0.$$

Variation with respect to $G(k)$ should reproduce the Dyson-Schwinger equation for $G(k)$:

$$G^{-1}(k) = D^{-1}(k) + \Pi(k)$$

Therefore $V_2(\phi, G)$ is constructed from

$$\Pi(k) = 2\frac{\delta V_2(\phi, G)}{\delta G(k)}.$$

Hartree truncation: Only tadpole contribution is retained to the self-energy

Lagrangean density, including a broad class of scalar models:

$$L = \frac{1}{2}[\partial_\mu \sigma_a \partial^\mu \sigma^a + \partial_\mu \pi_a \partial^\mu \pi_a - (\mu_S^2)_{ab}\sigma_a\sigma_b - (\mu_P^2)_{ab}\pi_a\pi_b]$$

$$-\frac{1}{3}F_{ab,cd}(\sigma_a\sigma_b\sigma_c\sigma_d + \pi_a\pi_b\pi_c\pi_d) - 2H_{ab,cd}\pi_a\pi_b\sigma_c\sigma_d.$$

Examples:

$O(N)$ model with 1 N-plet:

$$F_{abcd} = \frac{\lambda}{72N}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}), \qquad H_{abcd} = 0.$$

$O(N)$ model with 2 N-plets:

$$F^S_{abcd} = F^P_{abcd} = \frac{\lambda}{72N}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}), \qquad H_{abcd} = \frac{\lambda}{36N}\delta_{ab}\delta_{cd}.$$

$U(3) \times U(3)$ model for the meson nonet

$M = T^a(\sigma_a + i\pi_a), a = 0, .., 8$:

$$F_{abcd} = \frac{\lambda_1}{4}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + \frac{\lambda_2}{8}(d_{abn}d_{ncd} + d_{adn}d_{nbc} + d_{acn}d_{nbd})$$

$$H_{abcd} = \frac{\lambda_1}{4}\delta_{ab}\delta_{cd} + \frac{\lambda_2}{8}(d_{abn}d_{ncd} + f_{acn}f_{nbd} + f_{bcn}f_{nad})$$

2PI-Hartree effective potential with renormalised couplings plus counterterms in symmetry breaking σ-background

$$V_{full} = U(\bar\sigma_a) + V[\bar\sigma, S] + V_{ct}[\bar\sigma, S]$$

$$V[\bar\sigma_a, S_{ab}] = \frac{1}{2}\mathrm{Tr}\log S^{-1} + \frac{1}{2}\int_k (k^2\delta_{ab} - m^2_{R,ab})S_{ab} + Q^R_{ab,cd}\int_k S_{ab}(k)\int_p S_{cd}(p)$$

$$U(\bar\sigma_a) = \frac{1}{2}\mu^2_{R,ab}\bar\sigma_a\bar\sigma_b + \frac{1}{3}Q^R_{ab,cd}\bar\sigma_a\bar\sigma_b\bar\sigma_c\bar\sigma_d, \qquad m^2_{ab,R} = \mu^2_{R,ab} + 4Q^R_{ab,cd}\bar\sigma_c\bar\sigma_d$$

$$V_{ct}[\bar\sigma, S] = \delta U(\bar\sigma) + -\frac{1}{2}\int_k \delta m^2_{ab}S_{ab} + \delta Q_{ab,cd}\int_k S_{ab}(k)\int_p S_{cd}(p)$$

$$\delta U = \frac{1}{2}\left(\delta\mu^2_{ab} + \frac{2}{3}\delta\tilde{Q}_{ab,cd}\bar\sigma_c\bar\sigma_d\right)\bar\sigma_a\bar\sigma_b, \qquad \delta m^2_{ab} = \delta\mu^2_{ab} + 4\delta\hat{Q}_{ab,cd}\bar\sigma_c\bar\sigma_d$$

Note the compact notation!

$$S_{ab} = \{\langle\sigma_a\sigma_b\rangle, \langle\pi_a\pi_b\rangle\}, \qquad \mu^2_{ab} = \{(\mu^2_S)_{ab}, (\mu^2_P)_{ab}\},$$

$$Q^{11}_{abcd} = Q^{22}_{abcd} = F_{abcd}, \qquad Q^{12}_{abcd} = Q^{21}_{abcd} = H_{abcd}.$$

Three different 4-point counterterms are allowed to be introduced: $\delta Q, \delta\hat{Q}, \delta\tilde{Q}$!

The propagator (gap) equations

Variation with respect to $S_{cd}(k)$:

$$S^{-1}_{cd}(k) = k^2\delta_{cd} - m^2_{R,cd} - \delta m^2_{cd} + 4(Q^R_{abcd} + \delta Q_{abcd}) \int_p S_{ab}(p).$$

The self-energy matrix is momentum-independent: $S^{-1}_{ab} = k^2\delta_{ab} - M^2_{ab}$:

$$M^2_{cd} = m^2_{R,cd} + \delta m^2_{cd} - 4(Q^R_{abcd} + \delta Q_{abcd}) \int_p S_{ab}(p).$$

The resulting mass matrix is diagonalised by an orthogonal matrix O_{ci}:

$$\tilde{M}^2_i\delta_{ij} = O_{ci}m^2_{R,cd}O_{dj} + O_{ci}\delta m^2_{cd}O_{dj} - 4O_{ci}O_{dj}(Q^R_{ab,cd} + \delta Q_{ab,cd})O_{al}O_{bl}\int_k \frac{1}{k^2 - \tilde{M}^2_l}.$$

Separation of the divergent piece of the tadpole integral:

$$\int_k \frac{1}{k^2 - \tilde{M}^2_l} \equiv T(M^2_l) = T_{div}(M^2_l) + T_F(M^2_l), \qquad T_{div}(M^2_l) = \frac{\Lambda^2}{16\pi^2} + \tilde{M}^2_l B_D$$

$(B_D = \log(e\Lambda^2/M^2_0)/16\pi^2)$.

Renormalisation of the propagator (gap) equations I.

The renormalised matrix gap equation (finite parts of the above):

$$\tilde{M}_i^2 \delta_{ij} = O_{ci} m^2_{R,cd} O_{dj} - 4 O_{ci} O_{dj} Q^R_{ab,cd} O_{al} O_{bl} T_F(M_l^2).$$

Condition for the vanishing of the divergent pieces after the substitution of M_l^2 into the coeficient of the logarithmically divergent piece from the renormalised equation:

$$0 = \delta m^2_{cd} - 4(Q^R_{ab,cd} + \delta Q_{ab,cd}) \left(\frac{\Lambda^2}{16\pi^2} \delta_{ab} + m^2_{R,ab} B_D \right)$$

$$+16(Q^R_{ab,cd} + \delta Q_{ab,cd}) Q^R_{efab} O_{el} O_{fl} T_F(\tilde{M}_l^2) B_D - 4\delta Q_{ef,cd} O_{el} O_{fl} T_F(\tilde{M}_l^2).$$

Vanishing of the overall divergency (independent of T_F) and of the subdivergencies (the coefficients of each $T_F(M_l^2)$):

$$0 = \delta m^2_{cd} - 4(Q^R_{ab,cd} + \delta Q_{ab,cd}) \left(\frac{\Lambda^2}{16\pi^2} \delta_{ab} + m^2_{R,ab} B_D \right)$$

$$0 = 4 B_D (Q^R_{ab,cd} + \delta Q_{ab,cd}) Q^R_{ef,ab} - \delta Q_{ef,cd}.$$

Renormalisation of the propagator (gap) equations II.

The overall divergency is split into background independent and background dependent pieces

$$\delta\mu^2_{cd} = 4(Q^R_{ab,cd} + \delta Q_{ab,cd})\left(\frac{\Lambda^2}{16\pi^2}\delta_{ab} + \mu^2_{R,ab}B_D\right),$$

$$\delta\hat{Q}_{cdef}\bar{\sigma}_e\bar{\sigma}_f = 4B_D(Q^R_{ab,cd} + \delta Q_{ab,cd})Q^R_{abef}\bar{\sigma}_e\bar{\sigma}_f$$

The background dependent condition is equivalent to the previous condition if

$$\delta\hat{Q}_{cdef}\bar{\sigma}_e\bar{\sigma}_f = \delta Q_{cdef}\bar{\sigma}_e\bar{\sigma}_f$$

Compatibility with the equation of state

$$0 = \bar{\sigma}_b\left(\mu^2_{ab,R} + \tfrac{4}{3}(Q^R_{ab,cd} + \delta\tilde{Q}_{ab,cd})\bar{\sigma}_c\bar{\sigma}_d - 4(Q^R_{abcd} + \delta\hat{Q}_{abcd})\int_k S_{cd} + \delta\mu^2_{ab}\right)$$

The condition for the vanishing of the divergent piece remaining in the difference of the equation of state with the gap equation multiplied by $\bar{\sigma}_b$:

$$\left(\tfrac{1}{3}\delta\tilde{Q}_{abcd} - \delta Q_{abcd}\right)\bar{\sigma}_b\bar{\sigma}_c\bar{\sigma}_d = 0.$$

Example I: O(N) model with single N-plet

Two counterterm coupling is needed for solving the matrix equation of subdivergence cancellation:

$$\delta F_{abcd} = \frac{1}{24N}[\delta\lambda_A \delta_{ab}\delta_{cd} + \delta\lambda_B(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})]$$

Equations for the coefficents:

$$\delta\lambda_A = \frac{\lambda}{6N} B_D[(N+4)\lambda + (N+2)\delta\lambda_A + 2\delta\lambda_B],$$

$$\delta\lambda_B = \frac{\lambda}{3N} B_D[\lambda + \delta\lambda_B].$$

Parametrisation of the potential energy counterterm:

$$\delta\tilde{F}_{abcd} = \frac{\delta\tilde{\lambda}}{24N}(\delta_{ab}\delta_{bc} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})$$

leads to

$$\delta\tilde{\lambda} = \delta\lambda_A + 2\delta\lambda_B$$

Remarks:

1. The equations for $\delta\lambda_A, \delta\lambda_B$ coincide with those which can be derived with the method of iterative renormalisation (Blaizot, Iancu, Reinosa, 2004) where one looks for the self-energy in form of infinite series:

$$\Pi_{ab}(k) = \sum_n \Pi_{ab}^{(n)}(k), \qquad \delta\lambda_A = \sum_n \delta\lambda_A^{(n)}, \qquad \delta\lambda_B = \sum_n \delta\lambda_B^{(n)}$$

and solves the gap equations iteratavely.

2. When $N \to \infty$

$$\delta\lambda_A = -\frac{\lambda^2}{6} B_D \frac{1}{1 - \frac{\lambda}{6} B_D}, \qquad \delta\lambda_B \sim \mathcal{O}(1/N)$$

which leads to a unique quartic counter coupling $\delta\tilde{\lambda} = \delta\lambda_A$ and reproduces the exact result of the leading order large N analysis.

Example II: $O(N)$ model with 2 interacting N-plets

The gap equations for the mass matrix of the σ and π fields:

$$M^2_{S,cd} = m^2_{SR,cd} - 4(F^R_{ab,cd} + \delta F_{ab,cd}) \int S_{ab} - 4(H^R_{abcd} + \delta H_{abcd}) \int P_{ab} + \delta m^2_{S,cd},$$

$$M^2_{P,cd} = m^2_{PR,cd} - 4(F^R_{ab,cd} + \delta F_{ab,cd}) \int P_{ab} - 4(H^R_{abcd} + \delta H_{abcd}) \int S_{ab} + \delta m^2_{P,cd}$$

Subdivergence cancellation:

$$4B_D(F^R_{ab,cd} + \delta F_{ab,cd})F^R_{ef,ab} + 4B_D(H^R_{ab,cd} + \delta H_{ab,cd})H^R_{ef,ab} = \delta F_{ef,cd},$$

$$4B_D(F^R_{ab,cd} + \delta F_{ab,cd})H^R_{ef,ab} + 4B_D(H^R_{ab,cd} + \delta H_{ab,cd})F^R_{ef,ab} = \delta H_{ef,cd}$$

Parametrisation of the counter-coupling matrices:

$$\delta F_{ab,cd} = \frac{1}{24N}[\delta\lambda^F_A \delta_{ab}\delta_{cd} + \delta\lambda^F_B(\delta_{ac}\delta_{bd} + \delta_{ad}\delta bc)]$$

$$\delta H_{ab,cd} = \frac{1}{12N}\delta\lambda^H \delta_{ab}\delta_{cd}.$$

Example II: $O(N)$ model with 2 interacting N-plets

$$\delta\lambda_A^F = 4B_D\left(\frac{\lambda^2}{24N}(5N+4) + \frac{\lambda}{24N}(N+2)\delta\lambda_A^F + \frac{\lambda}{6}\delta\lambda^H\right),$$

$$\delta\lambda_B^F = 8B_D\frac{\lambda}{24N}(\lambda + \delta\lambda_B^F),$$

$$\delta\lambda^H = 4B_D\left(\frac{\lambda^2}{12N}(N+2) + \frac{\lambda}{24N}(N\delta\lambda_A^F + 2\delta\lambda_B^F) + \frac{\lambda}{24N}(N+2)\delta\lambda^H\right).$$

Large N limit:

$$\delta\lambda_A^F\left(1 - \frac{\lambda}{6}B_D\right) - \delta\lambda^H\frac{2\lambda}{3}B_D = \frac{5\lambda^2}{6}B_D,$$

$$-\delta\lambda_A^F\frac{\lambda}{6}B_D + \delta\lambda^H\left(1 - \frac{\lambda}{6}B_D\right) = \frac{\lambda^2}{3}B_D.$$

CONCLUSIONS (work to be done)

- Analysis of the $U(3) \times U(3)$ meson model
- Possible generalisation to any N and invetigaion of the $N \to \infty$ limit
- Solution of the renormalised gap equations for $N = 3$ and the quantitative comparison of the effect of the vacuum fluctuations on its thermodynamics.
- Going beyond the Hartree-approximation